

On Tamed Milstein Schemes of SDEs Driven by Lévy Noise

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Abstract

We extend the taming techniques developed in [3, 19] to construct explicit Milstein schemes that numerically approximate Lévy driven stochastic differential equations with super-linearly growing drift coefficients. The classical rate of convergence is recovered when the first derivative of the drift coefficient satisfies a polynomial Lipschitz condition.

Keywords: Explicit Milstein scheme, super-linear growth, rate of convergence, SDEs driven by Lévy noise.
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1 Introduction

The models incorporating event-driven uncertainties are increasingly becoming popular in finance, economics, engineering, medical sciences, ecology and many other branches of sciences. For example, in finance, often the stock price movements are suddenly and significantly influenced by market crashes, market booms, announcements made by central banks, changes in credit ratings, defaults, etc. The stochastic differential equations (SDEs) driven by Lévy noise are more realistic models to be used in such event-driven phenomena. Some of the references are [2], [16], [21] and references therein.

The Lévy driven SDEs rarely possess explicit solutions, that necessitates the development of numerical schemes to approximate their solutions. Recently, many explicit and implicit numerical schemes of Lévy driven SDEs have been studied and their convergence in strong as well as in weak sense have been proved. The interested reader may refer to [1], [4], [5], [7], [8], [13], [17] along with their references for a comprehensive discussion on numerous numerical schemes of different orders.

In [11], authors have shown that the efficient Euler schemes of SDEs (without jumps) with super-linearly growing drift coefficients diverge in \mathcal{L}^p -sense. Several numerical schemes based on taming techniques have been proposed in the past few years to overcome these difficulties. One could refer to [9], [10], [12], [19], [22], [24] and [20] for order 0.5 tamed Euler schemes and [23] and [24] for order 1.0 tamed Milstein schemes of SDEs (without jumps). For 0.5 order tamed schemes of SDEs driven by Lévy noise, the only reference available to us is [3]. The taming techniques proposed in this article can be extended to any desired order of convergence. However, for mathematical simplicity, we discuss order 1.0 tamed Milstein scheme of Lévy driven SDEs with super-linearly growing drift coefficients. It can also be concluded that our methodology and techniques can be extended to a tamed scheme of any order of convergence when drift coefficient grows super-linearly. Further, our calculations are motivated by the classical methodology (see also [3], [19]) and hence are more refined than those present in the literature. To the best of our knowledge, this is the first article on Milstein schemes of SDEs driven by Lévy noise when drift coefficient grows super-linearly.

We conclude this section by introducing some basic notations. We use $|a|$ to denote the Euclidean norm of a d -dimensional vector a , whereas $|A|$ for the Hilbert-Schmidt norm of a $d \times m$ matrix A . Also, the transpose of the matrix A is denoted by A^* . The inner product of two d -dimensional vectors x and y is

denoted by xy . The i th element of a d -dimensional vector is denoted by a^i , whereas $A^{(i,j)}$ and $A^{(j)}$ stand for (i,j) th element and j th column of a $d \times m$ matrix A respectively, for $i = 1, \dots, d$, $j = 1, \dots, m$. The notation $\lfloor x \rfloor$ stands for the integer part of a positive real number x . Finally, we use $K > 0$ to denote a generic constant, which is independent of n and varies from place to place.

2 Main Results

Let $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}, P)$ be a complete filtered probability space satisfying the usual conditions, i.e. the filtration is right continuous and \mathcal{F}_0 contains all P -null sets. Suppose that $T > 0$ is a fixed constant. We assume that $w_t := (w_t^i)_{i=1}^m$ is an \mathbb{R}^m -valued standard Wiener process. Also let (Z, \mathcal{Z}, ν) be σ -finite measure space and $N(dt, dz)$ be Poisson random measure defined on (Z, \mathcal{Z}, ν) with intensity ν satisfying $\nu(Z) < \infty$. We set $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$. Let $b(x)$ and $\sigma(x)$ be $\mathcal{B}(\mathbb{R}^d)$ -measurable functions with values in \mathbb{R}^d and $\mathbb{R}^{d \times m}$ respectively. Also, let $\gamma(x, z)$ be a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Z}$ -measurable function with values in \mathbb{R}^d . Moreover, $b(x)$, $\sigma(x)$ and $\gamma(x, z)$ are assumed to be twice differentiable functions in $x \in \mathbb{R}^d$. For the purpose of this article, we consider the following d -dimensional SDE,

$$x_t = \xi + \int_0^t b(x_s) ds + \int_0^t \sigma(x_s) dw_s + \int_0^t \int_Z \gamma(x_s, z) \tilde{N}(ds, dz), \quad (1)$$

almost surely for any $t \in [0, T]$, where ξ is an \mathcal{F}_0 -measurable random variable in \mathbb{R}^d .

Remark 1. In the right hand side of the above equation (1), we write x_t instead of x_{t-} in order to ease the notation. This does not cause any problem, since the compensator of the martingales driving the equation are continuous. This notational convenience shall be followed throughout this article.

We make the following assumptions.

A-1. For a fixed $p \geq 2$, $E|\xi|^p < \infty$.

A-2. There exists a constant $L > 0$ such that

$$(x - \bar{x})(b(x) - b(\bar{x})) \vee |\sigma(x) - \sigma(\bar{x})|^2 \vee \int_Z |\gamma(x, z) - \gamma(\bar{x}, z)|^2 \nu(dz) \leq L|x - \bar{x}|^2$$

for any $x, \bar{x} \in \mathbb{R}^d$.

A-3. There exists a constant $L > 0$ such that

$$\left| \frac{\partial \sigma^{(i,j)}(x)}{\partial x^k} - \frac{\partial \sigma^{(i,j)}(\bar{x})}{\partial x^k} \right|^2 \vee \int_Z \left| \frac{\partial \gamma^i(x, z)}{\partial x^k} - \frac{\partial \gamma^i(\bar{x}, z)}{\partial x^k} \right|^2 \nu(dz) \leq L|x - \bar{x}|^2$$

for any $x, \bar{x} \in \mathbb{R}^d$ and $i, k = 1, \dots, d$ and $j = 1, \dots, m$.

A-4. There exists a constant $L > 0$ such that

$$\int_Z |\gamma(x, z)|^p \nu(dz) \leq L(1 + |x|^p)$$

for any $x \in \mathbb{R}^d$.

A-5. There exist constants $L > 0$ and $\chi > 0$ such that

$$\left| \frac{\partial b^i(x)}{\partial x^k} - \frac{\partial b^i(\bar{x})}{\partial x^k} \right| \leq L(1 + |x|^\chi + |\bar{x}|^\chi)|x - \bar{x}|$$

for any $x, \bar{x} \in \mathbb{R}^d$, $i, k = 1, \dots, d$.

The main results of this article on the convergence of the newly proposed tamed Milstein scheme, which is formally introduced in Section 3, follows.

Theorem 1. *Let Assumptions A-1 to A-5 hold with $p \geq 6(\chi + 2)$. Then, the tamed Milstein scheme (3) converges to the solution of SDE (1) in \mathcal{L}^2 with rate of convergence arbitrarily close to 1, i.e.,*

$$\sup_{0 \leq t \leq T} E|x_t - x_t^n|^2 \leq Kn^{-\frac{2}{2+\delta}-1}$$

with $\delta \in (4/(p-2), 1)$.

Remark 2. If $E|\xi|^p < \infty$ for all $p > 0$, then one observes that the $\delta > 0$ appearing in Theorem 1 can be as small as one wishes and the obtained rate is arbitrarily close to 1.0 (which agrees with the classical results, see [17]).

The proof of Theorem 1 can be found in Section 4. When one considers only the continuous case, then the rate is improved as stated in the theorems below.

Theorem 2. *Let Assumptions A-1 to A-5 ($\gamma \equiv 0$) with $p \geq 6(\chi + 2)$ hold. Then, the tamed Milstein scheme (43) converges to the solution of SDE (42) in \mathcal{L}^q with rate 1, i.e.*

$$\sup_{0 \leq t \leq T} E|x_t - x_t^n|^q \leq Kn^{-q}$$

where $0 < q \leq \max(2, p/(3\chi + 6) - 2)$.

Theorem 3. *Let Assumptions A-1 to A-5 ($\gamma \equiv 0$) with $p \geq 6(\chi + 2)$ hold. Then, the tamed Milstein scheme (43) converges to the solution of SDE (42) in \mathcal{L}^q with rate 1, i.e.*

$$E \sup_{0 \leq t \leq T} |x_t - x_t^n|^q \leq Kn^{-q}$$

where $0 < q < \max(2, p/(3\chi + 6) - 2)$.

Remark 3. If one assumes that $E|\xi|^p < \infty$ is true for all $p > 0$ as assumed in [23], then Theorem 3 holds for any $q > 0$, which is in agreement with the findings of [23].

The proofs of Theorems [2, 3] as stated above are given in Section 5.

2.1 Preliminary Observations

The proof of the following lemma can be found in [15].

Lemma 1. *Let $r \geq 2$. There exists a constant K , depending only on r , such that for every real-valued, $\mathcal{B}([0, T]) \otimes \mathcal{Z}$ -measurable function g satisfying*

$$\int_0^T \int_Z |g_t(z)|^2 \nu(dz) dt < \infty$$

the following estimate holds,

$$\begin{aligned} E \sup_{0 \leq t \leq T} \left| \int_0^t \int_Z g_s(z) \tilde{N}(ds, dz) \right|^r &\leq KE \left(\int_0^T \int_Z |g_t(z)|^2 \nu(dz) dt \right)^{r/2} \\ &+ KE \int_0^T \int_Z |g_t(z)|^r \nu(dz) dt. \end{aligned}$$

It is known that if $1 \leq r \leq 2$, then the second term of the right hand side can be dropped.

Remark 4. Due to Assumption A-2, there exists a constant $L > 0$ such that

$$xb(x) \vee |\sigma(x)|^2 \vee \int_Z |\gamma(x, z)|^2 \nu(dz) \leq L(1 + |x|^2)$$

for any $x \in \mathbb{R}^d$.

The proof of the following lemma can be found in [3] and [21].

Lemma 2. *Let $b(x)$ be a continuous function in $x \in \mathbb{R}^d$. Also, suppose that Assumptions A-1, A-2 and A-4 are satisfied. Then, there exists a unique solution to SDE (1) and moreover*

$$E \sup_{0 \leq t \leq T} |x_t|^p \leq K,$$

where $K := K(L, T, p, E|\xi|^p)$.

Remark 5. Due to Assumptions A-2 and A-3, there exists a constant $L > 0$ such that

$$\left| \frac{\partial \sigma^{(i,j)}(x)}{\partial x^k} \right| \vee \left| \frac{\partial^2 \sigma^{(i,j)}(x)}{\partial x^k \partial x^u} \right| \vee \int_Z \left| \frac{\partial \gamma^i(x, z)}{\partial x^k} \right| \nu(dz) \vee \int_Z \left| \frac{\partial^2 \gamma^i(x, z)}{\partial x^k \partial x^u} \right| \nu(dz) \leq L$$

for any $x \in \mathbb{R}^d$, $i, k, u = 1, \dots, d$ and $j = 1, \dots, m$.

Remark 6. Due to Assumption A-5, there exist constants $L > 0$ and $\chi > 0$ such that

$$\left| \frac{\partial b^i(x)}{\partial x^k} \right| \leq L(1 + |x|^{\chi+1}) \quad \text{and} \quad \left| \frac{\partial^2 b^i(x)}{\partial x^j \partial x^k} \right| \leq L(1 + |x|^\chi)$$

for any $x \in \mathbb{R}^d$ and $i, j, k = 1, \dots, d$.

Remark 7. Due to Remark 6, there exist constants $L > 0$ and $\chi > 0$ such that

$$|b(x) - b(\bar{x})| \leq L(1 + |x|^{\chi+1} + |\bar{x}|^{\chi+1})|x - \bar{x}|$$

which further implies that

$$|b(x)| \leq L(1 + |x|^{\chi+2})$$

for any $x, \bar{x} \in \mathbb{R}^d$.

Remark 8. One notices that Assumption 3.1 in [23] implies Assumption A-3 of this article due to the fact that, for a function $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, the condition $\sup_{h_1, h_2 \in \mathbb{R}^d, |h_1| \leq 1, |h_2| \leq 1} \left| \sum_{i,j=1}^d \frac{\partial^2 f(x)}{\partial x^i \partial x^j} h_1^i h_2^j \right| \leq K$ implies that $\left| \frac{\partial^2 f^k(x)}{\partial x^i \partial x^j} \right| \leq K$ for every $i, j, k = 1, \dots, d$. One can make similar conclusion for Assumption A-5. Thus, the assumptions under consideration in this article are not stronger than those in [23].

Remark 9. For the practical implementation of the schemes (3) and (43) which have been introduced in this article, one requires commutative conditions on the coefficients, which have been discussed [14] and [17].

Remark 10. No comparison is made here with [24] as the main theorem for convergence in that article i.e. Theorem 2.2, p. 4 reproduces the corresponding theorem from [22], i.e. Theorem 2.1, p. 3137, which is known to contain an imprecise statement about the moments requirement for the \mathcal{L}^{2p} -convergence of their proposed numerical scheme.

The proof of the following lemma can be found in [18].

Lemma 3. *Let X be a positive, adapted right continuous process and A be a continuous increasing process such that*

$$E[X_\tau | \mathcal{F}_0] \leq E[A_\tau | \mathcal{F}_0]$$

for any bounded stopping time τ . Then for any $\varsigma \in (0, 1)$,

$$E[(X_\infty^*)^\varsigma] \leq \frac{2-\varsigma}{1-\varsigma} E[(A_\infty)^\varsigma]$$

where $X_\infty^* = \sup_{t \geq 0} X_t$.

3 Tamed Milstein Scheme

For every $n \in \mathbb{N}$, we propose the following form for taming the super-linearly growing drift coefficient of SDE (1),

$$\tilde{b}^n(x) = \frac{b(x)}{1 + n^{-\theta}|b(x)|^{2\theta}} \quad (2)$$

for any $\theta \geq \frac{1}{2}$, and $x \in \mathbb{R}^d$. One notes that \tilde{b}^n is an \mathbb{R}^d -valued function and its i -th element is denoted by $\tilde{b}^{n,i}$ for $i = 1, \dots, d$.

Remark 11. One observes that when $\theta = \frac{1}{2}$, then we obtain tamed Euler schemes similar to those discussed in [3] and [19]. In this article, we discuss a tamed Milstein scheme of SDE (1) by taking $\theta = 1$. It is important to note that by assigning different values to θ and appropriately including multiple stochastic integrals in the scheme, one could write a tamed scheme and then perform calculations similar to the methodology developed in this article to achieve an order θ . For the purpose of this article, we only discuss the case when $\theta = 1$ i.e. the tamed Milstein scheme of Lévy driven SDE with super-linear drift coefficient. From now onward, throughout this article, we take $\theta = 1$.

Remark 12. For every $n \in \mathbb{N}$, equation (2) implies

$$|\tilde{b}^n(x)| \leq \min(n^{1/2}, |b(x)|)$$

for any $x \in \mathbb{R}^d$ which is in the same spirit as in [3] and [19].

For every $n \in \mathbb{N}$, we propose the following tamed Milstein scheme,

$$x_t^n = \xi + \int_0^t \tilde{b}^n(x_{\kappa(n,s)}^n) ds + \int_0^t \tilde{\sigma}(x_{\kappa(n,s)}^n) dw_s + \int_0^t \int_Z \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2) \tilde{N}(ds, dz_2) \quad (3)$$

almost surely for any $t \in [0, T]$ where $\kappa(n, t) := \lfloor nt \rfloor / n$. The drift coefficient \tilde{b}^n of scheme (3) is defined in equation (2) whereas $\tilde{\sigma}$ and $\tilde{\gamma}$ are diffusion and jump coefficients of the scheme (3) which are defined below. For every $n \in \mathbb{N}$ and $s \in [0, T]$, the diffusion coefficient $\tilde{\sigma}(x_{\kappa(n,s)}^n)$ of scheme (3) is given by

$$\tilde{\sigma}(x_{\kappa(n,s)}^n) := \sigma(x_{\kappa(n,s)}^n) + \sigma_1(x_{\kappa(n,s)}^n) + \sigma_2(x_{\kappa(n,s)}^n) + \sigma_3(x_{\kappa(n,s)}^n) \quad (4)$$

where $\sigma_1(x_{\kappa(n,s)}^n)$, $\sigma_2(x_{\kappa(n,s)}^n)$ and $\sigma_3(x_{\kappa(n,s)}^n)$ are $d \times m$ matrices with their (i, k) -th elements given by

$$\begin{aligned} \sigma_1^{(i,k)}(x_{\kappa(n,s)}^n) &:= \sum_{j=1}^m \sum_{u=1}^d \int_{\kappa(n,s)}^s \sigma^{(u,j)}(x_{\kappa(n,r)}^n) \frac{\partial \sigma^{(i,k)}(x_{\kappa(n,r)}^n)}{\partial x^u} dw_r^j \\ \sigma_2^{(i,k)}(x_{\kappa(n,s)}^n) &:= \sum_{u=1}^d \int_{\kappa(n,s)}^s \int_Z \frac{\partial \sigma^{(i,k)}(x_{\kappa(n,r)}^n)}{\partial x^u} \gamma^u(x_{\kappa(n,r)}^n, z_1) \tilde{N}(dr, dz_1) \\ \sigma_3^{(i,k)}(x_{\kappa(n,s)}^n) &:= \int_{\kappa(n,s)}^s \int_Z \left(\sigma^{(i,k)}(x_{\kappa(n,r)}^n) + \gamma(x_{\kappa(n,r)}^n, z_1) - \sigma^{(i,k)}(x_{\kappa(n,r)}^n) \right. \\ &\quad \left. - \sum_{u=1}^d \frac{\partial \sigma^{(i,k)}(x_{\kappa(n,r)}^n)}{\partial x^u} \gamma^u(x_{\kappa(n,r)}^n, z_1) \right) N(dr, dz_1) \end{aligned} \quad (5)$$

respectively, for every $i = 1, \dots, d$, $k = 1, \dots, m$. Similarly, for every $n \in \mathbb{N}$, $s \in [0, T]$ and $z_2 \in Z$, the jump coefficient $\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)$ of scheme (3) is given by

$$\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2) := \gamma(x_{\kappa(n,s)}^n, z_2) + \gamma_1(x_{\kappa(n,s)}^n, z_2) + \gamma_2(x_{\kappa(n,s)}^n, z_2) + \gamma_3(x_{\kappa(n,s)}^n, z_2) \quad (6)$$

where $\gamma_1(x_{\kappa(n,s)}^n, z_2)$, $\gamma_2(x_{\kappa(n,s)}^n, z_2)$ and $\gamma_3(x_{\kappa(n,s)}^n, z_2)$ are d -dimensional vectors with their i th elements given by,

$$\begin{aligned}\gamma_1^i(x_{\kappa(n,s)}^n, z_2) &:= \sum_{j=1}^m \sum_{u=1}^d \int_{\kappa(n,s)}^s \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \sigma^{(u,j)}(x_{\kappa(n,r)}^n) dw_r^j \\ \gamma_2^i(x_{\kappa(n,s)}^n, z_2) &:= \sum_{u=1}^d \int_{\kappa(n,s)}^s \int_Z \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \gamma^u(x_{\kappa(n,r)}^n, z_1) \tilde{N}(dr, dz_1) \\ \gamma_3^i(x_{\kappa(n,s)}^n, z_2) &:= \int_{\kappa(n,s)}^s \int_Z \left(\gamma^i(x_{\kappa(n,r)}^n) + \gamma(x_{\kappa(n,r)}^n, z_1), z_2 \right) - \gamma^i(x_{\kappa(n,r)}^n, z_2) \\ &\quad - \sum_{u=1}^d \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \gamma^u(x_{\kappa(n,r)}^n, z_1) \Big) N(dr, dz_1)\end{aligned}$$

respectively, for every $i = 1, \dots, d$.

3.1 Moment Bounds

One observes that due to Remark 12, for a fixed $n \in \mathbb{N}$, the drift coefficient of the scheme (3) is bounded. Also, the diffusion and jump coefficients grow linearly due to Remark 4. Hence, one can refer to [17] to conclude that Assumptions A-1, A-2 and A-4 along with equation (2) imply that, for a fixed $n \in \mathbb{N}$,

$$E \sup_{0 \leq t \leq T} |x_t^n|^p < \infty. \quad (7)$$

Clearly, one can not claim at this stage that the bound is independent of n . However, it guarantees that all local martingales appearing henceforth are in fact true martingales and thus the use of stopping time arguments is avoided. Before proving the moment bounds of the scheme (3) in Lemma 11, one requires to show the following lemmas.

Lemma 4. *Let Assumption A-2 hold, then*

$$E|\sigma_1(x_{\kappa(n,s)}^n)|^p \leq Kn^{-\frac{p}{2}}(1 + E|x_{\kappa(n,s)}^n|^p),$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$, where $K := K(L, p, m, d)$.

Proof. First one writes,

$$\begin{aligned}E|\sigma_1(x_{\kappa(n,s)}^n)|^p &= E\left(\sum_{k=1}^m \sum_{i=1}^d |\sigma_1^{(i,k)}(x_{\kappa(n,s)}^n)|^2\right)^{\frac{p}{2}} \\ &\leq K \sum_{k,j=1}^m \sum_{i,u=1}^d E\left|\int_{\kappa(n,s)}^s \sigma^{(u,j)}(x_{\kappa(n,r)}^n) \frac{\partial \sigma^{(i,k)}(x_{\kappa(n,r)}^n)}{\partial x^u} dw_r^j\right|^p\end{aligned}$$

which on the application of an elementary inequality for stochastic integrals and Hölder's inequality yields

$$\begin{aligned}E|\sigma_1(x_{\kappa(n,s)}^n)|^p &\leq K \sum_{k,j=1}^m \sum_{i,u=1}^d E\left(\int_{\kappa(n,s)}^s \left|\sigma^{(u,j)}(x_{\kappa(n,r)}^n) \frac{\partial \sigma^{(i,k)}(x_{\kappa(n,r)}^n)}{\partial x^u}\right|^2 dr\right)^{\frac{p}{2}} \\ &\leq Kn^{-(\frac{p}{2}-1)} \sum_{k,j=1}^m \sum_{i,u=1}^d E \int_{\kappa(n,s)}^s |\sigma^{(u,j)}(x_{\kappa(n,r)}^n)|^p \left|\frac{\partial \sigma^{(i,k)}(x_{\kappa(n,r)}^n)}{\partial x^u}\right|^p dr\end{aligned}$$

and then due to Remarks [4, 5], one obtains

$$E|\sigma_1(x_{\kappa(n,s)}^n)|^p \leq K n^{-(\frac{p}{2}-1)} E \int_{\kappa(n,s)}^s (1 + |x_{\kappa(n,r)}^n|^p) dr = K n^{-\frac{p}{2}} (1 + E|x_{\kappa(n,s)}^n|^p)$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$. This completes the proof. \square

Lemma 5. *Let Assumptions A-2 and A-4 hold, then*

$$E|\sigma_2(x_{\kappa(n,s)}^n)|^p \leq K n^{-1} (1 + E|x_{\kappa(n,s)}^n|^p),$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$, where $K := K(L, p, m, d)$.

Proof. One notes that,

$$\begin{aligned} E|\sigma_2(x_{\kappa(n,s)}^n)|^p &= E \left(\sum_{i=1}^d \sum_{k=1}^m |\sigma_2^{(i,k)}(x_{\kappa(n,s)}^n)|^2 \right)^{\frac{p}{2}} \leq K \sum_{i=1}^d \sum_{k=1}^m E|\sigma_2^{(i,k)}(x_{\kappa(n,s)}^n)|^p \\ &\leq K \sum_{i=1}^d \sum_{k=1}^m \sum_{u=1}^d E \left| \int_{\kappa(n,s)}^s \int_Z \frac{\partial \sigma^{(i,k)}(x_{\kappa(n,r)}^n)}{\partial x^u} \gamma^u(x_{\kappa(n,r)}^n, z_1) \tilde{N}(dr, dz_1) \right|^p \end{aligned}$$

which due to Lemma 1 yields

$$\begin{aligned} &E|\sigma_2(x_{\kappa(n,s)}^n)|^p \\ &\leq K \sum_{i=1}^d \sum_{k=1}^m \sum_{u=1}^d E \left(\int_{\kappa(n,s)}^s \int_Z \left| \frac{\partial \sigma^{(i,k)}(x_{\kappa(n,r)}^n)}{\partial x^u} \gamma^u(x_{\kappa(n,r)}^n, z_1) \right|^2 \nu(dz_1) dr \right)^{\frac{p}{2}} \\ &+ K \sum_{i=1}^d \sum_{k=1}^m \sum_{u=1}^d E \int_{\kappa(n,s)}^s \int_Z \left| \frac{\partial \sigma^{(i,k)}(x_{\kappa(n,r)}^n)}{\partial x^u} \gamma^u(x_{\kappa(n,r)}^n, z_1) \right|^p \nu(dz_1) dr \end{aligned}$$

and then on using Remarks [4, 5] and Assumption A-4, one obtains

$$E|\sigma_2(x_{\kappa(n,s)}^n)|^p \leq K n^{-\frac{p}{2}} (1 + E|x_{\kappa(n,s)}^n|^p) + K n^{-1} (1 + E|x_{\kappa(n,s)}^n|^p)$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$. This finishes the proof. \square

Lemma 6. *Let Assumptions A-2 to A-4 hold, then*

$$E|\sigma_3(x_{\kappa(n,s)}^n)|^p \leq K n^{-1} (1 + E|x_{\kappa(n,s)}^n|^p),$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$, where $K := K(L, p, m, d)$.

Proof. By using Assumption A-2 and Remark 5, one obtains

$$\begin{aligned} E|\sigma_3(x_{\kappa(n,s)}^n)|^p &= E \left(\sum_{i=1}^d \sum_{k=1}^m |\sigma_3^{(i,k)}(x_{\kappa(n,s)}^n)|^2 \right)^{\frac{p}{2}} \\ &\leq K E \sum_{i=1}^d \sum_{k=1}^m \left| \int_{\kappa(n,s)}^s \int_Z \left(|\sigma^{(i,k)}(x_{\kappa(n,r)}^n) + \gamma(x_{\kappa(n,r)}^n, z_1)| - |\sigma^{(i,k)}(x_{\kappa(n,r)}^n)| \right. \right. \\ &\quad \left. \left. + \sum_{u=1}^d \left| \frac{\partial \sigma^{(i,k)}(x_{\kappa(n,r)}^n)}{\partial x^u} \right| |\gamma^u(x_{\kappa(n,r)}^n, z_1)| \right) N(dr, dz_1) \right|^p \\ &\leq K E \left(\int_{\kappa(n,s)}^s \int_Z |\gamma(x_{\kappa(n,r)}^n, z_1)| N(dr, dz_1) \right)^p \end{aligned}$$

which due to Lemma 1 gives

$$\begin{aligned}
E|\sigma_3(x_{\kappa(n,s)}^n)|^p &\leq KE \left(\int_{\kappa(n,s)}^s \int_Z |\gamma(x_{\kappa(n,r)}^n, z_1)| \tilde{N}(dr, dz_1) \right)^p \\
&\quad + KE \left(\int_{\kappa(n,s)}^s \int_Z |\gamma(x_{\kappa(n,r)}^n, z_1)| \nu(dz_1) dr \right)^p \\
&\leq KE \left(\int_{\kappa(n,s)}^s \int_Z |\gamma(x_{\kappa(n,r)}^n, z_1)|^2 \nu(dz_1) dr \right)^{\frac{p}{2}} \\
&\quad + KE \int_{\kappa(n,s)}^s \int_Z |\gamma(x_{\kappa(n,r)}^n, z_1)|^p \nu(dz_1) dr \\
&\quad + KE \left(\int_{\kappa(n,s)}^s \int_Z |\gamma(x_{\kappa(n,r)}^n, z_1)| \nu(dz_1) dr \right)^p
\end{aligned}$$

and then on applying Remark 4 and Assumption A-4, one obtains

$$\begin{aligned}
E|\sigma_3(x_{\kappa(n,s)}^n)|^p &\leq Kn^{-\frac{p}{2}} (1 + E|x_{\kappa(n,s)}^n|^p) + Kn^{-1} (1 + E|x_{\kappa(n,s)}^n|^p) \\
&\quad + Kn^{-p} (1 + E|x_{\kappa(n,s)}^n|^p)
\end{aligned}$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$. This completes the proof. \square

The following corollary is a consequence of Lemmas [4, 5, 6].

Corollary 1. *Let Assumptions A-2 to A-4 hold, then*

$$E|\tilde{\sigma}(x_{\kappa(n,s)}^n)|^p \leq K(1 + E|x_{\kappa(n,s)}^n|^p),$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$, where $K := K(L, p, m, d)$.

Lemma 7. *Let Assumptions A-2 and A-4 hold, then*

$$E \int_Z |\gamma_1(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) \leq Kn^{-\frac{p}{2}} (1 + E|x_{\kappa(n,s)}^n|^p),$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$, where $K := K(L, p, m, d)$.

Proof. One observes that

$$\begin{aligned}
E \int_Z |\gamma_1(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) &= E \int_Z \left(\sum_{i=1}^d |\gamma_1^i(x_{\kappa(n,s)}^n, z_2)|^2 \right)^{\frac{p}{2}} \nu(dz_2) \\
&\leq K \sum_{i=1}^d \int_Z \sum_{j=1}^m \sum_{u=1}^d E \left| \int_{\kappa(n,s)}^s \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \sigma^{(u,j)}(x_{\kappa(n,r)}^n) dw_r^j \right|^p \nu(dz_2)
\end{aligned}$$

which on using an elementary inequality for stochastic integrals and Hölder's inequality implies

$$\begin{aligned}
&E \int_Z |\gamma_1(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) \\
&\leq K \sum_{i,u=1}^d \sum_{j=1}^m \int_Z E \left(\int_{\kappa(n,s)}^s \left| \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \sigma^{(u,j)}(x_{\kappa(n,r)}^n) \right|^2 dr \right)^{\frac{p}{2}} \nu(dz_2) \\
&\leq Kn^{-(\frac{p}{2}-1)} \sum_{i,u=1}^d \sum_{j=1}^m \int_Z E \int_{\kappa(n,s)}^s \left| \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \right|^p |\sigma^{(u,j)}(x_{\kappa(n,r)}^n)|^p dr \nu(dz_2)
\end{aligned}$$

and then due to Remarks [4, 5], one obtains

$$E \int_Z |\gamma_1(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) \leq K n^{-\frac{p}{2}} (1 + E|x_{\kappa(n,s)}^n|^p)$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$. Hence the proof follows. \square

Lemma 8. *Let Assumptions A-2 and A-4 hold, then*

$$E \int_Z |\gamma_2(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) \leq K n^{-1} (1 + E|x_{\kappa(n,s)}^n|^p),$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$, where $K := K(L, p, m, d)$.

Proof. One observes that

$$\begin{aligned} E \int_Z |\gamma_2(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) &= E \int_Z \left(\sum_{i=1}^d |\gamma_2^i(x_{\kappa(n,s)}^n, z_2)|^2 \right)^{\frac{p}{2}} \nu(dz_2) \\ &\leq K \int_Z \sum_{i=1}^d \sum_{u=1}^d E \left| \int_{\kappa(n,s)}^s \int_Z \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \gamma^u(x_{\kappa(n,r)}^n, z_1) \tilde{N}(dr, dz_1) \right|^p \nu(dz_2) \end{aligned}$$

which due to Lemma 1 yields

$$\begin{aligned} &E \int_Z |\gamma_2(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) \\ &\leq K \int_Z \sum_{i,u=1}^d E \left(\int_{\kappa(n,s)}^s \int_Z \left| \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \gamma^u(x_{\kappa(n,r)}^n, z_1) \right|^2 \nu(dz_1) dr \right)^{\frac{p}{2}} \nu(dz_2) \\ &+ K \int_Z \sum_{i,u=1}^d E \int_{\kappa(n,s)}^s \int_Z \left| \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \gamma^u(x_{\kappa(n,r)}^n, z_1) \right|^p \nu(dz_1) dr \nu(dz_2) \end{aligned}$$

and then on using Remarks [4, 5] and Assumption A-4, one obtains,

$$E \int_Z |\gamma_2(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) \leq K n^{-\frac{p}{2}} (1 + E|x_{\kappa(n,s)}^n|^p) + K n^{-1} (1 + E|x_{\kappa(n,s)}^n|^p)$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$. Thus the proof finishes. \square

Lemma 9. *Let Assumptions A-2 to A-4 hold, then*

$$E \int_Z |\gamma_3(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) \leq K n^{-1} (1 + E|x_{\kappa(n,s)}^n|^p),$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$, where $K := K(L, p, m, d)$.

Proof. On using Assumption A-2 and Remark 5, one writes

$$\begin{aligned}
E \int_Z |\gamma_3(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) &\leq KE \int_Z \left(\sum_{i=1}^d |\gamma_3^i(x_{\kappa(n,s)}^n, z_2)|^2 \right)^{\frac{p}{2}} \nu(dz_2) \\
&= KE \int_Z \sum_{i=1}^d \left| \int_{\kappa(n,s)}^s \int_Z \left(\gamma^i(x_{\kappa(n,r)}^n + \gamma(x_{\kappa(n,r)}^n, z_1), z_2) - \gamma^i(x_{\kappa(n,r)}^n, z_2) \right. \right. \\
&\quad \left. \left. - \sum_{u=1}^d \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \gamma^u(x_{\kappa(n,r)}^n, z_1) \right) N(dr, dz_1) \right|^p \nu(dz_2) \\
&\leq KE \int_Z \sum_{i=1}^d \left(\int_{\kappa(n,s)}^s \int_Z \left(|\gamma^i(x_{\kappa(n,r)}^n + \gamma(x_{\kappa(n,r)}^n, z_1), z_2) - \gamma^i(x_{\kappa(n,r)}^n, z_2)| \right. \right. \\
&\quad \left. \left. + \sum_{u=1}^d \left| \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \right| |\gamma^u(x_{\kappa(n,r)}^n, z_1)| \right) N(dr, dz_1) \right)^p \nu(dz_2) \\
&\leq KE \int_Z \left(\int_{\kappa(n,s)}^s \int_Z |\gamma(x_{\kappa(n,r)}^n, z_1)| N(dr, dz_1) \right)^p \nu(dz_2)
\end{aligned}$$

which due to Assumption A-4, Lemma 1 and Remark 4 gives,

$$\begin{aligned}
&E \int_Z |\gamma_3(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) \\
&\leq K \int_Z E \left(\int_{\kappa(n,s)}^s \int_Z |\gamma(x_{\kappa(n,r)}^n, z_1)| \tilde{N}(dr, dz_1) \right)^p \nu(dz_2) \\
&+ K \int_Z E \left(\int_{\kappa(n,s)}^s \int_Z |\gamma(x_{\kappa(n,r)}^n, z_1)| \nu(dz_1) dr \right)^p \nu(dz_2) \\
&\leq Kn^{-\frac{p}{2}} (1 + E|x_{\kappa(n,s)}^n|^p) + Kn^{-1} (1 + E|x_{\kappa(n,s)}^n|^p) + Kn^{-p} (1 + E|x_{\kappa(n,s)}^n|^p)
\end{aligned}$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$. This completes the proof. \square

The following corollary is a consequence of Lemmas [7, 8, 9].

Corollary 2. *Let Assumptions A-2 to A-4 hold, then*

$$E \int_Z |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) \leq K(1 + E|x_{\kappa(n,s)}^n|^p),$$

for every $n \in \mathbb{N}$ and $s \in [0, T]$, where $K := K(L, p, m, d)$.

Lemma 10. *Let Assumptions A-2 to A-4 hold, then*

$$\int_0^u E|x_s^n - x_{\kappa(n,s)}^n|^p ds \leq Kn^{-1} \left(1 + \int_0^u E|x_{\kappa(n,s)}^n|^p ds \right),$$

for any $u \in [0, T]$, $n \in \mathbb{N}$, where $K := K(L, p, m, d)$.

Proof. Due to Hölder's inequality, Lemma 1 and Remark 12, one observes that

$$\begin{aligned}
E|x_s^n - x_{\kappa(n,s)}^n|^p &= E\left|\int_{\kappa(n,s)}^s \tilde{b}^n(x_{\kappa(n,r)}^n)dr + \int_{\kappa(n,s)}^s \tilde{\sigma}(x_{\kappa(n,r)}^n)dw_r\right. \\
&\quad \left. + \int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)\tilde{N}(dr, dz_2)\right|^p \\
&\leq KE\left|\int_{\kappa(n,s)}^s \tilde{b}^n(x_{\kappa(n,r)}^n)dr\right|^p + KE\left|\int_{\kappa(n,s)}^s \tilde{\sigma}(x_{\kappa(n,r)}^n)dw_r\right|^p \\
&\quad + KE\left|\int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)\tilde{N}(dr, dz_2)\right|^p \\
&\leq K|s - \kappa(n, s)|^{p-1}E\int_{\kappa(n,s)}^s |\tilde{b}^n(x_{\kappa(n,r)}^n)|^p dr \\
&\quad + K|s - \kappa(n, s)|^{\frac{p}{2}-1}\int_{\kappa(n,s)}^s E|\tilde{\sigma}(x_{\kappa(n,r)}^n)|^p dr \\
&\quad + K|s - \kappa(n, s)|^{\frac{p}{2}-1}\int_{\kappa(n,s)}^s E\left(\int_Z |\tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2)\right)^{\frac{p}{2}} dr \\
&\quad + K\int_{\kappa(n,s)}^s E\int_Z |\tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^p \nu(dz_2) dr
\end{aligned}$$

and then on using Corollaries [1, 2], one obtains

$$\begin{aligned}
E|x_s^n - x_{\kappa(n,s)}^n|^p &\leq Kn^{-\frac{p}{2}} + Kn^{-\frac{p}{2}}(1 + E|x_{\kappa(n,s)}^n|^p) \\
&\quad + Kn^{-\frac{p}{2}}(1 + E|x_{\kappa(n,s)}^n|^p) + Kn^{-1}(1 + E|x_{\kappa(n,s)}^n|^p)
\end{aligned}$$

for any $s \in [0, T]$, $n \in \mathbb{N}$. This completes the proof. \square

Lemma 11. *Let Assumptions A-1 to A-4 hold, then*

$$\sup_{n \in \mathbb{N}} E \sup_{0 \leq t \leq T} |x_t^n|^p \leq K,$$

where $K := K(L, T, p, m, d, E|\xi|^p)$.

Proof. By Itô's formula,

$$\begin{aligned}
|x_t^n|^p &= |\xi|^p + p \int_0^t |x_s^n|^{p-2} x_s^n \tilde{b}^n(x_{\kappa(n,s)}^n) ds + p \int_0^t |x_s^n|^{p-2} x_s^n \tilde{\sigma}(x_{\kappa(n,s)}^n) dw_s \\
&\quad + \frac{p(p-2)}{2} \int_0^t |x_s^n|^{p-4} |\tilde{\sigma}^*(x_{\kappa(n,s)}^n) x_s^n|^2 ds + \frac{p}{2} \int_0^t |x_s^n|^{p-2} |\tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 ds \\
&\quad + p \int_0^t \int_Z |x_s^n|^{p-2} x_s^n \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2) \tilde{N}(ds, dz_2) \\
&\quad + \int_0^t \int_Z (|x_s^n + \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^p - |x_s^n|^p - p|x_s^n|^{p-2} x_s^n \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)) N(ds, dz_2)
\end{aligned} \tag{8}$$

almost surely for any $t \in [0, T]$ and $n \in \mathbb{N}$. For estimating the second term on the right hand side of the above equation, one writes

$$x_s^n \tilde{b}^n(x_{\kappa(n,s)}^n) = (x_s^n - x_{\kappa(n,s)}^n) \tilde{b}^n(x_{\kappa(n,s)}^n) + x_{\kappa(n,s)}^n \tilde{b}^n(x_{\kappa(n,s)}^n)$$

which on using Remark 4 yields

$$\begin{aligned} x_s^n \tilde{b}^n(x_{\kappa(n,s)}^n) &\leq \left(\int_{\kappa(n,s)}^s \tilde{b}^n(x_{\kappa(n,r)}^n) dr + \int_{\kappa(n,s)}^s \tilde{\sigma}(x_{\kappa(n,r)}^n) dw_r \right. \\ &\quad \left. + \int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right) \tilde{b}^n(x_{\kappa(n,s)}^n) + L(1 + |x_{\kappa(n,s)}^n|^2) \end{aligned}$$

and then by using Remark 12 and Schwartz inequality, one obtains

$$\begin{aligned} x_s^n \tilde{b}^n(x_{\kappa(n,s)}^n) &\leq |\tilde{b}^n(x_{\kappa(n,s)}^n)| \int_{\kappa(n,s)}^s |\tilde{b}^n(x_{\kappa(n,r)}^n)| dr \\ &\quad + |\tilde{b}^n(x_{\kappa(n,s)}^n)| \left| \int_{\kappa(n,s)}^s \tilde{\sigma}(x_{\kappa(n,r)}^n) dw_r \right| \\ &\quad + |\tilde{b}^n(x_{\kappa(n,s)}^n)| \left| \int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right| + L(1 + |x_{\kappa(n,s)}^n|^2) \\ &\leq 1 + n^{\frac{1}{2}} \left| \int_{\kappa(n,s)}^s \tilde{\sigma}(x_{\kappa(n,r)}^n) dw_r \right| \\ &\quad + n^{\frac{1}{2}} \left| \int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right| + L(1 + |x_{\kappa(n,s)}^n|^2) \end{aligned}$$

almost surely for any $s \in [0, T]$ and $n \in \mathbb{N}$. Further, due to Young's inequality,

$$\begin{aligned} |x_s^n|^{p-2} x_s^n \tilde{b}^n(x_{\kappa(n,s)}^n) &\leq 1 + K|x_s^n|^p + K|x_{\kappa(n,s)}^n|^p + n^{\frac{p}{4}} \left| \int_{\kappa(n,s)}^s \tilde{\sigma}(x_{\kappa(n,r)}^n) dw_r \right|^{\frac{p}{2}} \\ &\quad + n^{\frac{1}{2}} |x_s^n|^{p-2} \left| \int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right| \end{aligned} \quad (9)$$

almost surely for any $s \in [0, T]$ and $n \in \mathbb{N}$.

Moreover, since the map $y \rightarrow |y|^p$ is of class C^2 , by the formula for the remainder,

$$|y_1 + y_2|^p - |y_1|^p - p|y_1|^{p-2} y_1 y_2 \leq K \int_0^1 |y_1 + qy_2|^{p-2} |y_2|^2 dq \leq K(|y_1|^{p-2} |y_2|^2 + |y_2|^p) \quad (10)$$

for any $y_1, y_2 \in \mathbb{R}^d$.

Hence, one first substitutes the estimates from (9) in (8) and uses (10) to estimate the last term of (8) which on taking suprema and expectation gives

$$\begin{aligned} E \sup_{0 \leq t \leq u} |x_t^n|^p &\leq E|\xi|^p + K + K \int_0^u E \sup_{0 \leq r \leq s} |x_r^n|^p ds + KE \int_0^u |x_s^n|^{p-2} |\tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 ds \\ &\quad + Kn^{\frac{p}{4}} \int_0^u E \left| \int_{\kappa(n,s)}^s \tilde{\sigma}(x_{\kappa(n,r)}^n) dw_r \right|^{\frac{p}{2}} ds \\ &\quad + pE \sup_{0 \leq t \leq u} \left| \int_0^t |x_s^n|^{p-2} x_s^n \tilde{\sigma}(x_{\kappa(n,s)}^n) dw_s \right| \\ &\quad + pE \sup_{0 \leq t \leq u} \left| \int_0^t \int_Z |x_s^n|^{p-2} x_s^n \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2) \tilde{N}(ds, dz_2) \right| \\ &\quad + E \int_0^u \int_Z \{ |x_s^n|^{p-2} |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^2 + |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^p \} N(ds, dz_2) \\ &\quad + Kn^{\frac{1}{2}} \int_0^u E |x_s^n|^{p-2} \left| \int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right| ds \\ &= C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7. \end{aligned} \quad (11)$$

Here, C_1 is given by

$$C_1 := E|\xi|^p + K + K \int_0^u E \sup_{0 \leq r \leq s} |x_r^n|^p ds$$

for any $u \in [0, T]$. Due to Young's inequality and Corollary 1, C_2 can be estimated by

$$\begin{aligned} C_2 &:= K E \int_0^u |x_s^n|^{p-2} |\tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 ds \leq K \int_0^u E |x_s^n|^p ds + K \int_0^u E |\tilde{\sigma}(x_{\kappa(n,s)}^n)|^p ds \\ &\leq K + K \int_0^u E |x_{\kappa(n,s)}^n|^p ds. \end{aligned} \quad (12)$$

For C_3 , one uses an elementary stochastic inequality and Corollary 1 to obtain

$$\begin{aligned} C_3 &:= K n^{\frac{p}{4}} \int_0^u E \left| \int_{\kappa(n,s)}^s \tilde{\sigma}(x_{\kappa(n,r)}^n) dw_r \right|^{\frac{p}{2}} ds \\ &\leq K n^{\frac{p}{4}} \int_0^u E \left(\int_{\kappa(n,s)}^s |\tilde{\sigma}(x_{\kappa(n,r)}^n)|^2 dr \right)^{\frac{p}{4}} ds \\ &\leq 1 + K n^{\frac{p}{2}} \int_0^u E \left(\int_{\kappa(n,s)}^s |\tilde{\sigma}(x_{\kappa(n,r)}^n)|^2 dr \right)^{\frac{p}{2}} ds \\ &\leq K n \int_0^u \int_{\kappa(n,s)}^s E |\tilde{\sigma}(x_{\kappa(n,r)}^n)|^p dr ds \\ &\leq K + K \int_0^u E |x_{\kappa(n,s)}^n|^p ds. \end{aligned} \quad (13)$$

Also, by using Burkholder-Gundy-Davis, Young's and Hölder's inequalities, C_4 can be estimated as

$$\begin{aligned} C_4 &:= p E \sup_{0 \leq t \leq u} \left| \int_0^t |x_s^n|^{p-2} x_s^n \tilde{\sigma}(x_{\kappa(n,s)}^n) dw_s \right| \leq K E \left(\int_0^u |x_s^n|^{2p-2} |\tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 ds \right)^{\frac{1}{2}} \\ &\leq K E \sup_{0 \leq t \leq u} |x_t^n|^{p-1} \left(\int_0^u |\tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K E \left(\int_0^u |\tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 ds \right)^{\frac{p}{2}} \\ &\leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K \int_0^u E |\tilde{\sigma}(x_{\kappa(n,s)}^n)|^p ds \end{aligned}$$

which due to Corollary 1 gives

$$C_4 \leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K + K \int_0^u E |x_{\kappa(n,s)}^n|^p ds. \quad (14)$$

Further, due to Lemma 1, Young's and Hölder's inequalities, one obtains

$$\begin{aligned} C_5 &:= E \sup_{0 \leq t \leq u} \int_0^t \int_Z |x_s^n|^{p-2} x_s^n \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2) |\tilde{N}(ds, dz_2)| \\ &\leq K E \int_0^u \left(\int_Z |x_s^n|^{2p-2} |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^2 \nu(dz_2) \right)^{\frac{1}{2}} ds \\ &\leq K E \sup_{0 \leq t \leq u} |x_t^n|^{p-1} \int_0^u \left(\int_Z |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^2 \nu(dz_2) \right)^{\frac{1}{2}} ds \\ &\leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K \int_0^u E \int_Z |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) ds \end{aligned}$$

which due to Corollary 2 gives

$$C_5 \leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K + K \int_0^u E |x_{\kappa(n,s)}^n|^p ds. \quad (15)$$

Moreover, C_6 can be estimated as

$$\begin{aligned} C_6 &:= E \int_0^u \int_Z (|x_s^n|^{p-2} |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^2 + |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^p) N(ds, dz_2) \\ &\leq E \int_0^u \int_Z (|x_s^n|^{p-2} |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^2 + |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^p) \nu(dz_2) ds \\ &\leq K E \int_0^u \int_Z |x_s^n|^{p-2} |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^2 \nu(dz_2) ds \\ &\quad + K E \int_0^u \int_Z |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) ds \\ &\leq K E \sup_{0 \leq t \leq u} |x_t^n|^{p-2} \int_0^u \int_Z |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^2 \nu(dz_2) ds \\ &\quad + K E \int_0^u \int_Z |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) ds \end{aligned}$$

which on the application of Young's inequality, Hölder's inequality and Corollary 2 implies

$$\begin{aligned} C_6 &\leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K \int_0^u E \int_Z |\tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^p \nu(dz_2) ds \\ &\leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K + K \int_0^u E |x_{\kappa(n,s)}^n|^p ds. \end{aligned} \quad (16)$$

Finally, for estimating C_7 , one uses the inequality $|x_s^n|^{p-2} \leq 2^{p-3} |x_{\kappa(n,s)}^n|^{p-2} + 2^{p-3} |x_s^n - x_{\kappa(n,s)}^n|^{p-2}$ to obtain,

$$\begin{aligned} C_7 &:= K n^{\frac{1}{2}} \int_0^u E |x_s^n|^{p-2} \left| \int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right| ds \\ &\leq K n^{\frac{1}{2}} \int_0^u E |x_{\kappa(n,s)}^n|^{p-2} \left| \int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right| ds \\ &\quad + K n^{\frac{1}{2}} \int_0^u E |x_s^n - x_{\kappa(n,s)}^n|^{p-2} \left| \int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right| ds \\ &=: C_{7a} + C_{7b}. \end{aligned} \quad (17)$$

To estimate C_{7a} , one uses Lemma 1 to write

$$\begin{aligned} C_{7a} &:= K n^{\frac{1}{2}} \int_0^u E \left| \int_{\kappa(n,s)}^s \int_Z |x_{\kappa(n,s)}^n|^{p-2} \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right| ds \\ &\leq K n^{\frac{1}{2}} \int_0^u E \left(\int_{\kappa(n,s)}^s \int_Z |x_{\kappa(n,s)}^n|^{2p-4} |\tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2) dr \right)^{\frac{1}{2}} ds \end{aligned}$$

which on using Young's and Hölder's inequalities gives

$$\begin{aligned} C_{7a} &\leq K E \sup_{0 \leq t \leq u} |x_t^n|^{p-2} n^{\frac{1}{2}} \int_0^u \left(\int_{\kappa(n,s)}^s \int_Z |\tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2) dr \right)^{\frac{1}{2}} ds \\ &\leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K n^{\frac{p}{4}} E \int_0^u \left(\int_{\kappa(n,s)}^s \int_Z |\tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2) dr ds \right)^{\frac{p}{4}} \\ &\leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + 1 + K n^{\frac{p}{2}} E \int_0^u \left(\int_{\kappa(n,s)}^s \int_Z |\tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2) dr ds \right)^{\frac{p}{2}} \\ &\leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + 1 + K n \int_0^u \int_{\kappa(n,s)}^s E \int_Z |\tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^p \nu(dz_2) dr ds \end{aligned}$$

and then using Corollary 2,

$$C_{7a} \leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K + K \int_0^u E |x_{\kappa(n,s)}^n|^p ds. \quad (18)$$

Further, to estimate C_{7b} , one observes that due to Young's inequality,

$$\begin{aligned} C_{7b} &:= K n^{\frac{1}{2}} \int_0^u E |x_s^n - x_{\kappa(n,s)}^n|^{p-2} \left| \int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right| ds \\ &\leq K n^{\frac{1}{2}} \int_0^u E |x_s^n - x_{\kappa(n,s)}^n|^p ds \\ &\quad + K n^{\frac{1}{2}} \int_0^u E \left| \int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right|^{\frac{p}{2}} ds \\ &\leq K n^{\frac{1}{2}} \int_0^u E |x_s^n - x_{\kappa(n,s)}^n|^p ds \\ &\quad + 1 + K n \int_0^u E \left| \int_{\kappa(n,s)}^s \int_Z \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right|^p ds \end{aligned}$$

which on using Lemmas [1, 10] gives

$$\begin{aligned} C_{7b} &\leq K n^{-\frac{1}{2}} \left(1 + \int_0^u E |x_{\kappa(n,s)}^n|^p ds \right) \\ &\quad + 1 + K n \int_0^u E \left(\int_{\kappa(n,s)}^s \int_Z |\tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2) dr \right)^{\frac{p}{2}} ds \\ &\quad + K n \int_0^u E \int_{\kappa(n,s)}^s \int_Z |\tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^p \nu(dz_2) dr ds \\ &\leq K + K \int_0^u E |x_{\kappa(n,s)}^n|^p ds \\ &\quad + K n^{2-\frac{p}{2}} \int_0^u \int_{\kappa(n,s)}^s E \int_Z |\tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^p \nu(dz_2) dr ds \\ &\quad + K n \int_0^u \int_{\kappa(n,s)}^s E \int_Z |\tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^{\frac{p}{2}} \nu(dz_2) dr ds \end{aligned}$$

and finally due to Corollary 2, one obtains

$$C_{7b} \leq K + K \int_0^u E |x_{\kappa(n,s)}^n|^p ds. \quad (19)$$

Thus, on using estimates from (18) and (19) in (17), one has

$$C_7 \leq \frac{1}{8} E \sup_{0 \leq t \leq u} |x_t^n|^p + K + K \int_0^u E |x_{\kappa(n,s)}^n|^p ds. \quad (20)$$

Hence, on substituting estimates from (12), (13), (14), (15), (16) and (20) in (11), one obtains due to (7)

$$E \sup_{0 \leq t \leq u} |x_t^n|^p \leq \frac{1}{2} E \sup_{0 \leq t \leq u} |x_t^n|^p + E |\xi|^p + K + K \int_0^u E \sup_{0 \leq r \leq s} |x_r^n|^p ds < \infty$$

which on the application of Gronwall's lemma completes the proof. \square

4 Rate of Convergence

In what follows, we assume that $6(\chi + 2) \leq p$ and $\delta \in (4/(p - 2), 1)$.

Lemma 12. *Let Assumptions A-1 to A-4 hold, then*

$$\sup_{0 \leq t \leq T} E|x_t^n - x_{\kappa(n,t)}^n|^r \leq Kn^{-1}$$

for any $2 \leq r \leq p$.

Proof. The proof follows due to Lemmas [10, 11]. □

Lemma 13. *Let Assumptions A-1 to A-5 hold, then*

$$\sup_{0 \leq t \leq T} E|\sigma(x_{\kappa(n,t)}^n)|^2 |x_t^n - x_{\kappa(n,t)}^n|^2 \leq Kn^{-1}.$$

Proof. By Remark 4 and Hölder's inequality,

$$\begin{aligned} E|\sigma(x_{\kappa(n,t)}^n)|^2 |x_t^n - x_{\kappa(n,t)}^n|^2 &\leq KE \left| \int_{\kappa(n,t)}^t (1 + |x_{\kappa(n,t)}^n|^2) \tilde{b}^n(x_{\kappa(n,r)}^n) dr \right|^2 \\ &\quad + KE \left| \int_{\kappa(n,t)}^t (1 + |x_{\kappa(n,t)}^n|^2) \tilde{\sigma}(x_{\kappa(n,r)}^n) dw_r \right|^2 \\ &\quad + KE \left| \int_{\kappa(n,t)}^t \int_Z (1 + |x_{\kappa(n,t)}^n|^2) \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) \right|^2 \\ &\leq Kn^{-1} E \int_{\kappa(n,t)}^t (1 + |x_{\kappa(n,t)}^n|^2)^2 |\tilde{b}^n(x_{\kappa(n,r)}^n)|^2 dr \\ &\quad + KE \int_{\kappa(n,t)}^t (1 + |x_{\kappa(n,t)}^n|^2)^2 |\tilde{\sigma}(x_{\kappa(n,r)}^n)|^2 dr \\ &\quad + KE \int_{\kappa(n,t)}^t \int_Z (1 + |x_{\kappa(n,t)}^n|^2)^2 |\tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2) dr. \end{aligned}$$

Then, on further application of Hölder's inequality along with Remark 7 and Corollaries [1, 2], one completes the proof. □

Lemma 14. *Let Assumptions A-1 to A-5 hold, then*

$$\sup_{0 \leq t \leq T} E \int_Z |\gamma(x_{\kappa(n,t)}^n, z_2)|^2 |x_t^n - x_{\kappa(n,t)}^n|^2 \nu(dz_2) \leq Kn^{-1}.$$

Proof. The proof follows the same arguments as used in Lemma 13. □

Lemma 15. *Consider equation (2) and let Assumptions A-1 to A-5 be satisfied, then*

$$\sup_{0 \leq t \leq T} E|b(x_t^n) - \tilde{b}^n(x_t^n)|^2 \leq Kn^{-2}.$$

Proof. The proof immediately follows due to equation (2), Remarks [12, 7] and Lemma 11. □

Lemma 16. *Let Assumptions A-1 to A-5 hold, then*

$$\sup_{0 \leq t \leq T} E \int_Z |\gamma(x_t^n, z_2) - \tilde{\gamma}(x_{\kappa(n,t)}^n, z_2)|^2 \nu(dz_2) \leq Kn^{-2}.$$

Proof. By Itô's formula,

$$\begin{aligned}
\gamma^i(x_t^n, z_2) &= \gamma^i(x_{\kappa(n,t)}^n, z_2) + \sum_{u=1}^d \int_{\kappa(n,t)}^t \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \tilde{b}^{n,u}(x_{\kappa(n,r)}^n) dr \\
&+ \frac{1}{2} \sum_{u_1, u_2=1}^d \sum_{l_1=1}^m \int_{\kappa(n,t)}^t \frac{\partial^2 \gamma^i(x_r^n, z_2)}{\partial x^{u_1} \partial x^{u_2}} \tilde{\sigma}^{(u_1, l_1)}(x_{\kappa(n,r)}^n) \tilde{\sigma}^{(u_2, l_1)}(x_{\kappa(n,r)}^n) dr \\
&+ \sum_{u=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \tilde{\sigma}^{(u, j)}(x_{\kappa(n,r)}^n) dw_r^j \\
&+ \sum_{u=1}^d \int_{\kappa(n,t)}^t \int_Z \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \tilde{\gamma}^u(x_{\kappa(n,r)}^n, z_1) \tilde{N}(dr, dz_1) \\
&+ \int_{\kappa(n,t)}^t \int_Z \left(\gamma^i(x_r^n + \tilde{\gamma}(x_{\kappa(n,r)}^n, z_1), z_2) - \gamma^i(x_r^n, z_2) \right. \\
&\quad \left. - \sum_{u=1}^d \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \tilde{\gamma}^u(x_{\kappa(n,r)}^n, z_1) \right) N(dr, dz_1).
\end{aligned} \tag{21}$$

Then, on taking expectation of square of difference between (21) and (6), One obtains

$$\begin{aligned}
& E \int_Z |\gamma^i(x_t^n, z_2) - \tilde{\gamma}^i(x_{\kappa(n,t)}^n, z_2)|^2 \nu(dz_2) \\
& \leq K \int_Z E \left| \sum_{u=1}^d \int_{\kappa(n,t)}^t \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \tilde{b}^{n,u}(x_{\kappa(n,r)}^n) dr \right|^2 \nu(dz_2) \\
& + K \int_Z E \left| \sum_{u_1, u_2=1}^d \sum_{l_1=1}^m \int_{\kappa(n,t)}^t \frac{\partial^2 \gamma^i(x_r^n, z_2)}{\partial x^{u_1} \partial x^{u_2}} \tilde{\sigma}^{(u_1, l_1)}(x_{\kappa(n,r)}^n) \tilde{\sigma}^{(u_2, l_1)}(x_{\kappa(n,r)}^n) dr \right|^2 \nu(dz_2) \\
& + K \int_Z E \left| \sum_{u=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left(\frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} - \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \right) \sigma^{(u,j)}(x_{\kappa(n,r)}^n) dw_r^j \right|^2 \nu(dz_2) \\
& + K \int_Z E \left| \sum_{u=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \sigma_1^{(u,j)}(x_{\kappa(n,r)}^n) dw_r^j \right|^2 \nu(dz_2) \\
& + K \int_Z E \left| \sum_{u=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \sigma_2^{(u,j)}(x_{\kappa(n,r)}^n) dw_r^j \right|^2 \nu(dz_2) \\
& + K \int_Z E \left| \sum_{u=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \sigma_3^{(u,j)}(x_{\kappa(n,r)}^n) dw_r^j \right|^2 \nu(dz_2) \\
& + K \int_Z E \left| \sum_{u=1}^d \int_{\kappa(n,t)}^t \int_Z \left[\frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} - \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \right] \gamma^u(x_{\kappa(n,r)}^n, z_1) \tilde{N}(dr, dz_1) \right|^2 \nu(dz_2) \\
& + K \int_Z E \left| \sum_{u=1}^d \int_{\kappa(n,t)}^t \int_Z \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \gamma_1^u(x_{\kappa(n,r)}^n, z_1) \tilde{N}(dr, dz_1) \right|^2 \nu(dz_2) \\
& + K \int_Z E \left| \sum_{u=1}^d \int_{\kappa(n,t)}^t \int_Z \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \gamma_2^u(x_{\kappa(n,r)}^n, z_1) \tilde{N}(dr, dz_1) \right|^2 \nu(dz_2) \\
& + K \int_Z E \left| \sum_{u=1}^d \int_{\kappa(n,t)}^t \int_Z \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \gamma_3^u(x_{\kappa(n,r)}^n, z_1) \tilde{N}(dr, dz_1) \right|^2 \nu(dz_2) \\
& + K \int_Z E \left[\int_{\kappa(n,t)}^t \int_Z \left(|\gamma^i(x_r^n + \tilde{\gamma}(x_{\kappa(n,r)}^n, z_1), z_2) - \gamma^i(x_{\kappa(n,r)}^n + \gamma(x_{\kappa(n,r)}^n, z_1), z_2)| \right. \right. \\
& + |\gamma^i(x_r^n, z_2) - \gamma^i(x_{\kappa(n,r)}^n, z_2)| + \sum_{u=1}^d \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} - \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \right| |\gamma^u(x_{\kappa(n,r)}^n, z_1)| \\
& + \sum_{u=1}^d \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right| |\gamma_1^u(x_{\kappa(n,r)}^n, z_1)| + \sum_{u=1}^d \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right| |\gamma_2^u(x_{\kappa(n,r)}^n, z_1)| \\
& \left. \left. + \sum_{u=1}^d \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right| |\gamma_3^u(x_{\kappa(n,r)}^n, z_1)| \right) N(dr, dz_1) \right]^2 \nu(dz_2)
\end{aligned}$$

which due to Hölder's inequality, Fubini Theorem and Lemma 1 gives,

$$\begin{aligned}
& E \int_Z |\gamma^i(x_t^n, z_2) - \tilde{\gamma}^i(x_{\kappa(n,t)}^n, z_2)|^2 \nu(dz_2) \\
& \leq K n^{-1} E \sum_{u=1}^d \int_{\kappa(n,t)}^t \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\tilde{b}^{n,u}(x_{\kappa(n,r)}^n)|^2 dr \\
& + K n^{-1} E \sum_{u_1, u_2=1}^d \sum_{l_1=1}^m \int_{\kappa(n,t)}^t \left(\int_Z \left| \frac{\partial^2 \gamma^i(x_r^n, z_2)}{\partial x^{u_1} \partial x^{u_2}} \right|^2 \nu(dz_2) \right) |\tilde{\sigma}^{(u_1, l_1)}(x_{\kappa(n,r)}^n) \tilde{\sigma}^{(u_2, l_1)}(x_{\kappa(n,r)}^n)|^2 dr \\
& + K E \sum_{u=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} - \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\sigma^{(u,j)}(x_{\kappa(n,r)}^n)|^2 dr \\
& + K E \sum_{u=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\sigma_1^{(u,j)}(x_{\kappa(n,r)}^n)|^2 dr \\
& + K E \sum_{u=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\sigma_2^{(u,j)}(x_{\kappa(n,r)}^n)|^2 dr \\
& + K E \sum_{u=1}^d \sum_{j=1}^m \int_{\kappa(n,t)}^t \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\sigma_3^{(u,j)}(x_{\kappa(n,r)}^n)|^2 dr \\
& + K E \sum_{u=1}^d \int_{\kappa(n,t)}^t \int_Z \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} - \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\gamma^u(x_{\kappa(n,r)}^n, z_1)|^2 \nu(dz_1) dr \\
& + K E \sum_{u=1}^d \int_{\kappa(n,t)}^t \int_Z \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\gamma_1^u(x_{\kappa(n,r)}^n, z_1)|^2 \nu(dz_1) dr \\
& + K E \sum_{u=1}^d \int_{\kappa(n,t)}^t \int_Z \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\gamma_2^u(x_{\kappa(n,r)}^n, z_1)|^2 \nu(dz_1) dr \\
& + K E \sum_{u=1}^d \int_{\kappa(n,t)}^t \int_Z \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\gamma_3^u(x_{\kappa(n,r)}^n, z_1)|^2 \nu(dz_1) dr \\
& + K E \int_{\kappa(n,t)}^t \int_Z \left[\left(\int_Z |\gamma^i(x_r^n + \tilde{\gamma}(x_{\kappa(n,r)}^n, z_1), z_2) - \gamma^i(x_{\kappa(n,r)}^n + \gamma(x_{\kappa(n,r)}^n, z_1), z_2)|^2 \nu(dz_2) \right) \right. \\
& \quad \left. + \left(\int_Z |\gamma^i(x_r^n, z_2) - \gamma^i(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2) \right) \right. \\
& \quad + \sum_{u=1}^d \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} - \frac{\partial \gamma^i(x_{\kappa(n,r)}^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\gamma^u(x_{\kappa(n,r)}^n, z_1)|^2 \\
& \quad + \sum_{u=1}^d \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\gamma_1^u(x_{\kappa(n,r)}^n, z_1)|^2 \\
& \quad + \sum_{u=1}^d \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\gamma_2^u(x_{\kappa(n,r)}^n, z_1)|^2 \\
& \quad \left. + \sum_{u=1}^d \left(\int_Z \left| \frac{\partial \gamma^i(x_r^n, z_2)}{\partial x^u} \right|^2 \nu(dz_2) \right) |\gamma_3^u(x_{\kappa(n,r)}^n, z_1)|^2 \right] \nu(dz_1) dr.
\end{aligned}$$

Thus, due to Assumption A-3 and Remark 5,

$$\begin{aligned}
& E \int_Z |\gamma(x_t^n, z_2) - \tilde{\gamma}(x_{\kappa(n,t)}^n, z_2)|^2 \nu(dz_2) \\
& \leq Kn^{-1} E \int_{\kappa(n,t)}^t |\tilde{b}^n(x_{\kappa(n,r)}^n)|^2 dr + Kn^{-1} E \int_{\kappa(n,t)}^t |\tilde{\sigma}(x_{\kappa(n,r)}^n)|^2 dr \\
& + KE \int_{\kappa(n,t)}^t |\sigma(x_{\kappa(n,r)}^n)|^2 |x_r^n - x_{\kappa(n,r)}^n|^2 dr \\
& + KE \int_{\kappa(n,t)}^t |\sigma_1(x_{\kappa(n,r)}^n)|^2 dr + KE \int_{\kappa(n,t)}^t |\sigma_2(x_{\kappa(n,r)}^n)|^2 dr \\
& + KE \int_{\kappa(n,t)}^t |\sigma_3(x_{\kappa(n,r)}^n)|^2 dr \\
& + K \int_{\kappa(n,t)}^t \int_Z |x_r^n - x_{\kappa(n,r)}^n|^2 |\gamma(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2) dr \\
& + KE \int_{\kappa(n,t)}^t \int_Z |\gamma_1(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2) dr \\
& + KE \int_{\kappa(n,t)}^t \int_Z |\gamma_2(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2) dr \\
& + KE \int_{\kappa(n,t)}^t \int_Z |\gamma_3(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2) dr
\end{aligned} \tag{22}$$

for any $t \in [0, T]$. Finally, one completes the proof by using Remarks [4, 12, 7], Lemmas [4, 5, 6, 7, 8, 9, 11, 12, 13, 14] along with Corollaries [1, 2]. \square

Lemma 17. *Let Assumptions A-1 to A-5 hold, then*

$$\sup_{0 \leq t \leq T} E |\sigma(x_t^n) - \tilde{\sigma}(x_{\kappa(n,t)}^n)|^2 \leq Kn^{-2}.$$

Proof. The lemma can be proved by adopting similar arguments as used in the proof of Lemma 16. \square

Before proceeding further, let us define $e_t^n := x_t - x_t^n$ i.e.

$$\begin{aligned}
e_t^n &= \int_0^t (b(x_s) - \tilde{b}^n(x_{\kappa(n,s)}^n)) ds + \int_0^t (\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}^n)) dw_s \\
&+ \int_0^t \int_Z (\gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)) \tilde{N}(ds, dz_2)
\end{aligned}$$

almost surely for any $t \in [0, T]$ and $n \in \mathbb{N}$. Also, the notation $e_t^{n,k}$ stands for the k -th element of e_t^n for every $k = 1, \dots, d$, $t \in [0, T]$ and $n \in \mathbb{N}$.

Lemma 18. *Let Assumptions A-1 to A-5 hold, then*

$$E \int_0^t e_s^n (b(x_s^n) - b(x_{\kappa(n,s)}^n)) ds \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^2 ds + Kn^{-\frac{2}{2+\delta}-1}$$

for any $t \in [0, T]$.

Proof. First, one applies Itô's formula to obtain $b^k(x_s^n) - b^k(x_{\kappa(n,s)}^n)$ and then writes the following,

$$\begin{aligned}
E \int_0^t e_s^n (b(x_s^n) - b(x_{\kappa(n,s)}^n)) ds &= E \sum_{k=1}^d \int_0^t e_s^{n,k} \{b^k(x_s^n) - b^k(x_{\kappa(n,s)}^n)\} ds \\
&= E \sum_{k,i=1}^d \int_0^t e_s^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{b}^{n,i}(x_{\kappa(n,r)}^n) dr ds \\
&\quad + \frac{1}{2} \sum_{k,i,j=1}^d E \int_0^t e_s^{n,k} \int_{\kappa(n,s)}^s \frac{\partial^2 b^k(x_r^n)}{\partial x^i \partial x^j} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) \tilde{\sigma}^{(j,l)}(x_{\kappa(n,r)}^n) dr ds \\
&\quad + \sum_{k,i=1}^d E \int_0^t e_s^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&\quad + \sum_{k,i=1}^d E \int_0^t e_s^{n,k} \int_{\kappa(n,s)}^s \int_Z \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\gamma}^i(x_{\kappa(n,r)}^n, z_2) \tilde{N}(dr, dz_2) ds \\
&\quad + \sum_{k=1}^d E \int_0^t e_s^{n,k} \int_{\kappa(n,s)}^s \int_Z \left(b^k(x_r^n) + \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2) \right. \\
&\quad \quad \left. - b^k(x_r^n) - \sum_{i=1}^d \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\gamma}^i(x_{\kappa(n,r)}^n, z_2) \right) N(dr, dz_2) ds \\
&=: F_1 + F_2 + F_3 + F_4 + F_5
\end{aligned} \tag{23}$$

for any $t \in [0, T]$. By using Young's and Hölder's inequalities, F_1 can be estimated as

$$\begin{aligned}
F_1 &:= \sum_{k,i=1}^d E \int_0^t e_s^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{b}^{n,i}(x_{\kappa(n,r)}^n) dr ds \\
&\leq K E \int_0^t |e_s^n|^2 ds + K n^{-1} \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s \left| \frac{\partial b^k(x_r^n)}{\partial x^i} \right|^2 |\tilde{b}^{n,i}(x_{\kappa(n,r)}^n)|^2 dr ds
\end{aligned}$$

and then due to Remarks [12, 6, 7] and Lemma 11, one obtains

$$F_1 \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^2 ds + K n^{-2}. \tag{24}$$

Similarly, F_2 can be estimated by using Young's and Hölder's inequalities along with Remarks [4, 6], Corollary 1 and Lemma 11 as,

$$\begin{aligned}
F_2 &:= \frac{1}{2} \sum_{k,i,j=1}^d E \int_0^t e_s^{n,k} \int_{\kappa(n,s)}^s \frac{\partial^2 b^k(x_r^n)}{\partial x^i \partial x^j} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) \tilde{\sigma}^{(j,l)}(x_{\kappa(n,r)}^n) dr ds \\
&\leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^2 ds + K n^{-2}.
\end{aligned} \tag{25}$$

For the estimation of F_3 , one uses

$$\begin{aligned} e_s^{n,k} = & e_{\kappa(n,s)}^{n,k} + \int_{\kappa(n,s)}^s (b^k(x_r) - \tilde{b}^{n,k}(x_{\kappa(n,r)}^n)) dr \\ & + \sum_{v=1}^m \int_{\kappa(n,s)}^s (\sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n)) dw_r^v \\ & + \int_{\kappa(n,s)}^s \int_Z (\gamma^k(x_r, z_2) - \tilde{\gamma}^k(x_{\kappa(n,r)}^n, z_2)) \tilde{N}(dr, dz_2) \end{aligned}$$

to obtain the following,

$$\begin{aligned} F_3 := & K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t e_s^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\ \leq & K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\ & + K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \int_{\kappa(n,s)}^s (b^k(x_r) - \tilde{b}^{n,k}(x_{\kappa(n,r)}^n)) dr \\ & \quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\ & + K \sum_{k,i=1}^d \sum_{l,v=1}^m E \int_0^t \int_{\kappa(n,s)}^s (\sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n)) dw_r^v \\ & \quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\ & + K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \int_{\kappa(n,s)}^s \int_Z (\gamma^k(x_r, z_2) - \tilde{\gamma}^k(x_{\kappa(n,r)}^n, z_2)) \tilde{N}(dr, dz_2) \\ & \quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\ =: & F_{31} + F_{32} + F_{33} + F_{34}. \end{aligned} \tag{26}$$

Here, F_{31} is given by

$$\begin{aligned} F_{31} := & K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\ = & K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\ & + K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_1^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\ & + K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_2^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\ & + K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_3^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \end{aligned}$$

which on the application of Young's inequality gives,

$$\begin{aligned}
F_{31} &\leq KE \int_0^t |e_{\kappa(n,s)}^n|^2 ds + K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_1^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^2 ds \\
&\quad + K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_2^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^2 ds \\
&\quad + K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_3^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^2 ds
\end{aligned}$$

and then on using an elementary inequality of stochastic integrals along with Remark 6, one obtains

$$\begin{aligned}
F_{31} &\leq KE \int_0^t |e_{\kappa(n,s)}^n|^2 ds + KE \int_0^t \int_{\kappa(n,s)}^s (1 + |x_r^n|^\chi)^2 |\sigma_1(x_{\kappa(n,r)}^n)|^2 dr ds \\
&\quad + KE \int_0^t \int_{\kappa(n,s)}^s (1 + |x_r^n|^\chi)^2 |\sigma_2(x_{\kappa(n,r)}^n)|^2 dr ds \\
&\quad + KE \int_0^t \int_{\kappa(n,s)}^s (1 + |x_r^n|^\chi)^2 |\sigma_3(x_{\kappa(n,r)}^n)|^2 dr ds.
\end{aligned}$$

Also, by using Hölder's inequality and Lemmas [4, 5, 6, 11],

$$\begin{aligned}
F_{31} &\leq KE \int_0^t |e_{\kappa(n,s)}^n|^2 ds \\
&\quad + K \int_0^t \int_{\kappa(n,s)}^s (E(1 + |x_r^n|^\chi)^{\frac{2(2+\delta)}{\delta}})^{\frac{\delta}{2+\delta}} (E|\sigma_1(x_{\kappa(n,r)}^n)|^{2+\delta})^{\frac{2}{2+\delta}} dr ds \\
&\quad + K \int_0^t \int_{\kappa(n,s)}^s (E(1 + |x_r^n|^\chi)^{\frac{2(2+\delta)}{\delta}})^{\frac{\delta}{2+\delta}} (E|\sigma_2(x_{\kappa(n,r)}^n)|^{2+\delta})^{\frac{2}{2+\delta}} dr ds \\
&\quad + K \int_0^t \int_{\kappa(n,s)}^s (E(1 + |x_r^n|^\chi)^{\frac{2(2+\delta)}{\delta}})^{\frac{\delta}{2+\delta}} (E|\sigma_3(x_{\kappa(n,r)}^n)|^{2+\delta})^{\frac{2}{2+\delta}} dr ds \\
&\leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^2 ds + Kn^{-\frac{2}{2+\delta}-1}.
\end{aligned} \tag{27}$$

Further, due to Remark 7,

$$\begin{aligned}
|b^k(x_r) - \tilde{b}^{n,k}(x_{\kappa(n,r)}^n)| &\leq L(1 + |x_r|^\chi + |x_r^n|^\chi) |x_r - x_r^n| \\
&\quad + L(1 + |x_r^n|^\chi + |x_{\kappa(n,r)}^n|^\chi) |x_r^n - x_{\kappa(n,r)}^n| \\
&\quad + |b^k(x_{\kappa(n,r)}^n) - \tilde{b}^{n,k}(x_{\kappa(n,r)}^n)|
\end{aligned} \tag{28}$$

for any $r \in [0, T]$, $k = 1, \dots, d$, which on substituting in F_{32} gives

$$\begin{aligned}
F_{32} &:= K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \int_{\kappa(n,s)}^s (b^k(x_r) - \tilde{b}^{n,k}(x_{\kappa(n,r)}^n)) dr \\
&\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&\leq K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \int_{\kappa(n,s)}^s (1 + |x_r|^\chi + |x_r^n|^\chi) |x_r - x_r^n| dr \\
&\quad \times \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right| ds \\
&\quad + K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \int_{\kappa(n,s)}^s (1 + |x_r^n|^\chi + |x_{\kappa(n,r)}^n|^\chi) |x_r^n - x_{\kappa(n,r)}^n| dr \\
&\quad \times \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right| ds \\
&\quad + K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \int_{\kappa(n,s)}^s |b^k(x_{\kappa(n,r)}^n) - \tilde{b}^{n,k}(x_{\kappa(n,r)}^n)| dr \\
&\quad \times \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right| ds.
\end{aligned}$$

Now, one uses Hölder's inequality with exponent $\sqrt{2}$ and $2 + \sqrt{2}$ to obtain,

$$\begin{aligned}
F_{32} &\leq K \sum_{k,i=1}^d \sum_{l=1}^m \int_0^t \left(n^{-\sqrt{2}+1} E \int_{\kappa(n,s)}^s (1 + |x_r|^\chi + |x_r^n|^\chi)^{\sqrt{2}} |x_r - x_r^n|^{\sqrt{2}} dr \right)^{\frac{1}{\sqrt{2}}} \\
&\quad \times \left(E \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^{2+\sqrt{2}} \right)^{\frac{1}{2+\sqrt{2}}} ds \\
&\quad + K \sum_{k,i=1}^d \sum_{l=1}^m \int_0^t \left(n^{-\sqrt{2}+1} E \int_{\kappa(n,s)}^s (1 + |x_r^n|^\chi + |x_{\kappa(n,r)}^n|^\chi)^{\sqrt{2}} |x_r^n - x_{\kappa(n,r)}^n|^{\sqrt{2}} dr \right)^{\frac{1}{\sqrt{2}}} \\
&\quad \times \left(E \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^{2+\sqrt{2}} \right)^{\frac{1}{2+\sqrt{2}}} ds \\
&\quad + K \sum_{k,i=1}^d \sum_{l=1}^m \int_0^t \left(n^{-1} E \int_{\kappa(n,s)}^s |b^k(x_{\kappa(n,r)}^n) - \tilde{b}^{n,k}(x_{\kappa(n,r)}^n)|^2 dr \right. \\
&\quad \times \left. E \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^2 \right)^{\frac{1}{2}} ds
\end{aligned}$$

which on the application of Hölder's inequality along with Remark 6, Lemmas [2, 11, 15] and Corollary 1

yields the following,

$$\begin{aligned}
F_{32} &\leq K n^{-\frac{1}{2}} \int_0^t \left(n^{-\sqrt{2}+1} \int_{\kappa(n,s)}^s (E(1 + |x_r|^\chi + |x_r^n|^\chi)^{\sqrt{2}(2+\sqrt{2})})^{\frac{1}{2+\sqrt{2}}} \right. \\
&\quad \times (E|e_r^n|^2)^{\frac{1}{\sqrt{2}}} dr \Big)^{\frac{1}{\sqrt{2}}} ds \\
&\quad + K n^{-\frac{1}{2}} \int_0^t \left(n^{-\sqrt{2}+1} \int_{\kappa(n,s)}^s (E(1 + |x_r^n|^\chi + |x_{\kappa(n,r)}^n|^\chi)^{\sqrt{2}(2+\sqrt{2})})^{\frac{1}{2+\sqrt{2}}} \right. \\
&\quad \times (E|x_r^n - x_{\kappa(n,r)}^n|^2)^{\frac{1}{\sqrt{2}}} dr \Big)^{\frac{1}{\sqrt{2}}} ds + K n^{-\frac{5}{2}} \\
&\leq K n^{-\frac{3}{2}} \int_0^t \left(\sup_{0 \leq r \leq s} E|e_r^n|^2 \right)^{\frac{1}{2}} ds + K n^{-2} + K n^{-\frac{5}{2}}.
\end{aligned}$$

Thus, due to Young's inequality, one obtains

$$F_{32} \leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^2 ds + K n^{-2}. \quad (29)$$

Again, for any $r \in [0, T]$, $k = 1, \dots, d$ and $v = 1, \dots, m$, one uses

$$\begin{aligned}
\sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n) &= (\sigma^{(k,v)}(x_r) - \sigma^{(k,v)}(x_r^n)) \\
&\quad + (\sigma^{(k,v)}(x_r^n) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n))
\end{aligned} \quad (30)$$

to express F_{33} as below,

$$\begin{aligned}
F_{33} &:= K \sum_{k,i=1}^d \sum_{l,v=1}^m E \int_0^t \int_{\kappa(n,s)}^s (\sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n)) dw_r^v \\
&\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&= K \sum_{k,i=1}^d \sum_{l,v=1}^m E \int_0^t \int_{\kappa(n,s)}^s (\sigma^{(k,v)}(x_r) - \sigma^{(k,v)}(x_r^n)) dw_r^v \\
&\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&\quad + K \sum_{k,i=1}^d \sum_{l,v=1}^m E \int_0^t \int_{\kappa(n,s)}^s (\sigma^{(k,v)}(x_r^n) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n)) dw_r^v \\
&\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds.
\end{aligned}$$

Further, Hölder's inequality implies

$$\begin{aligned}
F_{33} &\leq K \sum_{k,i=1}^d \sum_{l,v=1}^m \int_0^t \left(E \left| \int_{\kappa(n,s)}^s (\sigma^{(k,v)}(x_r) - \sigma^{(k,v)}(x_r^n)) dw_r^v \right|^2 \right. \\
&\quad \times E \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^2 \Big)^{\frac{1}{2}} ds \\
&\quad + K \sum_{k,i=1}^d \sum_{l,v=1}^m \int_0^t \left(E \left| \int_{\kappa(n,s)}^s (\sigma^{(k,v)}(x_r^n) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n)) dw_r^v \right|^2 \right. \\
&\quad \times E \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^2 \Big)^{\frac{1}{2}} ds
\end{aligned}$$

which again due to Hölder's inequality, Remark 6, Lemma 11 and Corollary 1 yields the following,

$$\begin{aligned} F_{33} &\leq Kn^{-\frac{1}{2}} \int_0^t \left(E \int_{\kappa(n,s)}^s |\sigma(x_r) - \sigma(x_r^n)|^2 dr \right)^{\frac{1}{2}} ds \\ &\quad + Kn^{-\frac{1}{2}} \int_0^t \left(E \int_{\kappa(n,s)}^s |\sigma(x_r^n) - \tilde{\sigma}(x_{\kappa(n,r)}^n)|^2 dr \right)^{\frac{1}{2}} ds. \end{aligned}$$

and then Assumption A-2 and Lemma 17 gives

$$F_{33} \leq Kn^{-1} \int_0^t \left(\sup_{0 \leq r \leq s} E |e_r^n|^2 \right)^{\frac{1}{2}} ds + Kn^{-2}.$$

Thus, on the application of Young's inequality, one obtains

$$F_{33} \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^2 ds + Kn^{-2}. \quad (31)$$

Finally, for any $r \in [0, T]$ $k = 1, \dots, d$ and $z_2 \in Z$, one uses

$$\gamma^k(x_r, z_2) - \tilde{\gamma}^k(x_{\kappa(n,r)}^n, z_2) = (\gamma^k(x_r, z_2) - \gamma^k(x_r^n, z_2)) + (\gamma^k(x_r^n, z_2) - \tilde{\gamma}^k(x_{\kappa(n,r)}^n, z_2))$$

to express F_{34} as the following

$$\begin{aligned} F_{34} &:= K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \int_{\kappa(n,s)}^s \int_Z (\gamma^k(x_r, z_2) - \tilde{\gamma}^k(x_{\kappa(n,r)}^n, z_2)) \tilde{N}(dr, dz_2) \\ &\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\ &= K \sum_{k,i=1}^d \sum_{l=1}^m \int_0^t E \int_{\kappa(n,s)}^s \int_Z (\gamma^k(x_r, z_2) - \gamma^k(x_r^n, z_2)) \tilde{N}(dr, dz_2) \\ &\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\ &\quad + K \sum_{k,i=1}^d \sum_{l=1}^m \int_0^t E \int_{\kappa(n,s)}^s \int_Z (\gamma^k(x_r^n, z_2) - \tilde{\gamma}^k(x_{\kappa(n,r)}^n, z_2)) \tilde{N}(dr, dz_2) \\ &\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \end{aligned}$$

which due to Hölder's inequality gives,

$$\begin{aligned} F_{34} &\leq K \sum_{k,i=1}^d \sum_{l=1}^m \int_0^t \left(E \left| \int_{\kappa(n,s)}^s \int_Z (\gamma^k(x_r, z_2) - \gamma^k(x_r^n, z_2)) \tilde{N}(dr, dz_2) \right|^2 \right. \\ &\quad \times \left. E \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^2 \right)^{\frac{1}{2}} ds \\ &\quad + K \sum_{k,i=1}^d \sum_{l=1}^m \int_0^t \left(E \left| \int_{\kappa(n,s)}^s \int_Z (\gamma^k(x_r^n, z_2) - \tilde{\gamma}^k(x_{\kappa(n,r)}^n, z_2)) \tilde{N}(dr, dz_2) \right|^2 \right. \\ &\quad \times \left. E \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^2 \right)^{\frac{1}{2}} ds. \end{aligned}$$

Further, the application of Assumption A-2, Remark 6, Lemmas [1, 16] and Corollary 1 gives

$$\begin{aligned}
F_{34} &\leq K n^{-\frac{1}{2}} \int_0^t \left(E \int_{\kappa(n,s)}^s \int_Z |\gamma(x_r, z_2) - \gamma(x_r^n, z_2)|^2 \nu(dz_2) dr \right)^{\frac{1}{2}} ds \\
&\quad + K n^{-\frac{1}{2}} \int_0^t \left(E \int_{\kappa(n,s)}^s \int_Z |\gamma(x_r^n, z_2) - \tilde{\gamma}(x_{\kappa(n,r)}^n, z_2)|^2 \nu(dz_2) dr \right)^{\frac{1}{2}} ds \\
&\leq K n^{-1} \int_0^t \left(\sup_{0 \leq r \leq s} E |e_r^n|^2 \right)^{\frac{1}{2}} ds + K n^{-2}.
\end{aligned}$$

Thus, on the application of Young's inequality, one obtains

$$F_{34} \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^2 ds + K n^{-2}. \quad (32)$$

Thus substituting the estimates from (27), (29), (31) and (32) in (26), one obtains

$$F_3 \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^2 ds + K n^{-\frac{2}{2+\delta}-1}. \quad (33)$$

By adopting the same approach as followed in the estimation of F_3 , one could estimate F_4 and F_5 as

$$F_4 \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^2 ds + K n^{-\frac{2}{2+\delta}-1}, \quad (34)$$

$$F_5 \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^2 ds + K n^{-\frac{2}{2+\delta}-1}. \quad (35)$$

Finally, combining estimates from (24), (25), (33), (34) and (35) in equation (23) completes the proof. \square

Proof of Theorem 1. By Itô's formula,

$$\begin{aligned}
|e_t^n|^2 &= 2 \int_0^t e_s^n (b(x_s) - \tilde{b}^n(x_{\kappa(n,s)}^n)) ds + 2 \int_0^t e_s^n (\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}^n)) dw_s \\
&\quad + \int_0^t |\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 ds + 2 \int_0^t \int_Z e_s^n (\gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)) \tilde{N}(ds, dz_2) \\
&\quad + \int_0^t \int_Z (|e_s^n + \gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^2 \\
&\quad \quad - |e_s^n|^2 - 2e_s^n (\gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2))) N(ds, dz_2)
\end{aligned}$$

almost surely for any $t \in [0, T]$, which on taking expectation implies

$$\begin{aligned}
E |e_t^n|^2 &= 2E \int_0^t e_s^n \{b(x_s) - \tilde{b}^n(x_{\kappa(n,s)}^n)\} ds + E \int_0^t |\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 ds \\
&\quad + E \int_0^t \int_Z (|e_s^n + \gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^2 \\
&\quad \quad - |e_s^n|^2 - 2e_s^n \{\gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)\}) \nu(dz_2) ds \\
&=: W_1 + W_2 + W_3
\end{aligned} \quad (36)$$

for any $t \in [0, T]$.

Now, for estimating W_1 , one uses the following,

$$\begin{aligned}
e_s^n (b(x_s) - \tilde{b}^n(x_{\kappa(n,s)}^n)) &= e_s^n (b(x_s) - b(x_s^n)) + e_s^n (b(x_s^n) - b(x_{\kappa(n,s)}^n)) \\
&\quad + e_s^n (b(x_{\kappa(n,s)}^n) - \tilde{b}^n(x_{\kappa(n,s)}^n))
\end{aligned}$$

which on using Assumption A-2 and Young's inequality gives

$$\begin{aligned} e_s^n(b(x_s) - \tilde{b}^n(x_{\kappa(n,s)}^n)) &\leq K|e_s^n|^2 + e_s^n(b(x_s^n) - b(x_{\kappa(n,s)}^n)) \\ &\quad + K|b(x_{\kappa(n,s)}^n) - \tilde{b}^n(x_{\kappa(n,s)}^n)|^2 \end{aligned} \quad (37)$$

for any $s \in [0, T]$. Thus, W_1 can be estimated by

$$\begin{aligned} W_1 &:= 2E \int_0^t e_s^n(b(x_s) - \tilde{b}^n(x_{\kappa(n,s)}^n)) ds \\ &\leq KE \int_0^t |e_s^n|^2 ds + 2E \int_0^t e_s^n(b(x_s^n) - b(x_{\kappa(n,s)}^n)) ds \\ &\quad + KE \int_0^t |b(x_{\kappa(n,s)}^n) - \tilde{b}^n(x_{\kappa(n,s)}^n)|^2 ds \end{aligned}$$

which on the application of Lemma [15, 18] implies

$$W_1 \leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^2 ds + Kn^{-\frac{2}{2+\delta}-1}. \quad (38)$$

Further, for estimating W_2 , one writes,

$$\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}^n) = \sigma(x_s) - \sigma(x_s^n) + \sigma(x_s^n) - \tilde{\sigma}(x_{\kappa(n,s)}^n)$$

which on using Assumption A-2 yields,

$$|\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 \leq K|e_s^n|^2 + 2|\sigma(x_s^n) - \tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 \quad (39)$$

for any $s \in [0, T]$. Thus, due to Lemma 17, one obtains

$$W_2 := E \int_0^t |\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 ds \leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^2 ds + Kn^{-2}. \quad (40)$$

Finally, for estimating W_3 , one uses equation (10) to write

$$\begin{aligned} W_3 &:= E \int_0^t \int_Z (|e_s^n + \gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^2 - |e_s^n|^2 \\ &\quad - 2e_s^n(\gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2))) \nu(dz_2) ds \\ &\leq KE \int_0^t \int_Z |\gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^2 \nu(dz_2) ds \end{aligned}$$

and then applying the following splitting

$$\begin{aligned} \gamma(x_s, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2) &= \gamma(x_s, z_2) - \gamma(x_s^n, z_2) \\ &\quad + \gamma(x_s^n, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2), \end{aligned}$$

one could write

$$\begin{aligned} W_3 &\leq KE \int_0^t \int_Z |\gamma(x_s, z_2) - \gamma(x_s^n, z_2)|^2 \nu(dz_2) ds \\ &\quad + KE \int_0^t \int_Z |\gamma(x_s^n, z_2) - \tilde{\gamma}(x_{\kappa(n,s)}^n, z_2)|^2 \nu(dz_2) ds \end{aligned}$$

which due to Assumption A-2 and Lemma 16 yields

$$W_3 \leq K \int_0^t \sup_{0 \leq r \leq s} E|e_r^n|^2 ds + Kn^{-2}. \quad (41)$$

Thus by substituting estimates from (38), (40) and (41) in (36), one obtains,

$$\sup_{0 \leq t \leq u} E|e_t^n|^2 \leq K \int_0^u \sup_{0 \leq r \leq s} E|e_r^n|^2 ds + Kn^{-\frac{2}{2+\delta}-1} < \infty$$

for any $u \in [0, T]$. The Gronwall's inequality completes the proof. \square

5 Continuous Case

In this section, the case when $\nu \equiv 0$ or $\gamma \equiv 0$ in SDE (1) is discussed, i.e. one considers the following SDE,

$$x_t = \xi + \int_0^t b(x_s) ds + \int_0^t \sigma(x_s) dw_s \quad (42)$$

almost surely for any $t \in [0, T]$, where ξ is an \mathcal{F}_0 -measurable random variable in \mathbb{R}^d . Further, the scheme (3) is replaced by the following,

$$x_t^n = \xi + \int_0^t \tilde{b}^n(x_{\kappa(n,s)}^n) ds + \int_0^t \tilde{\sigma}(x_{\kappa(n,s)}^n) dw_s, \quad (43)$$

almost surely for any $t \in [0, T]$. The drift coefficient \tilde{b}^n in scheme (43) is given by equation (2) and the diffusion coefficient $\tilde{\sigma}$ is defined as

$$\tilde{\sigma}(x_{\kappa(n,s)}^n) := \sigma(x_{\kappa(n,s)}^n) + \sigma_1(x_{\kappa(n,s)}^n) \quad (44)$$

where σ_1 is given by (5). Also, Assumptions A-2 to A-4 hold since $\gamma \equiv 0$. This implies that Lemma 2 holds under Assumptions A-1 and A-2 with $b(x)$ being a continuous function in $x \in \mathbb{R}^d$. Further, Lemma 4 and Corollary 1 hold under Assumptions A-2 and A-3 while Lemma 11 holds under Assumptions A-1 to A-3. One also notes that Lemma 5 to Lemma 9 and Corollary 2 are not required for this section since $\gamma \equiv 0$.

We now proceed for the derivation of the rate of convergence of scheme (43) and prove that this is same as that obtained by [23]. One can notice in the following calculations that a rate of convergence is achieved for any $q \leq p/(3\chi + 6)$ when either $p/(3\chi + 6) \geq 4$ or $p/(3\chi + 6) = 2$.

Lemma 19. *Consider Remark 12 and let Assumptions A-1 to A-3 (with $\gamma \equiv 0$) hold, then*

$$\sup_{0 \leq t \leq T} E|x_t^n - x_{\kappa(n,t)}^n|^q \leq Kn^{-\frac{q}{2}}.$$

Proof. This follows due to Lemmas [10, 11]. \square

Lemma 20. *Consider equation (2) and let Assumptions A-1 to A-3 (with $\gamma \equiv 0$) be satisfied, then*

$$\sup_{0 \leq t \leq T} E|b(x_t^n) - \tilde{b}^n(x_t^n)|^q dt \leq Kn^{-q}.$$

Proof. This is an immediate consequence of equation (2), Remark 4 and Lemma 11. \square

Lemma 21. *Let Assumptions A-1 to A-3 (with $\gamma \equiv 0$) hold, then*

$$\sup_{0 \leq t \leq T} E|\sigma(x_t^n) - \tilde{\sigma}(x_{\kappa(n,t)}^n)|^q \leq Kn^{-q}.$$

Proof. One uses Itô's formula to estimate the difference $\sigma^{(k,v)}(x_t^n) - \sigma^{(k,v)}(x_{\kappa(n,t)}^n)$ and (44) to obtain the following,

$$\begin{aligned}
E|\sigma(x_t^n) - \tilde{\sigma}(x_{\kappa(n,t)}^n)|^q &= E\left(\sum_{k=1}^d \sum_{v=1}^m |\sigma^{(k,v)}(x_t^n) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,t)}^n)|^2\right)^{\frac{q}{2}} \\
&\leq K \sum_{u,k=1}^d \sum_{v=1}^m E\left|\int_{\kappa(n,t)}^t \frac{\partial \sigma^{(k,v)}(x_r^n)}{\partial x^u} \tilde{b}^{n,u}(x_{\kappa(n,r)}^n) dr\right|^q \\
&+ K \sum_{u,k=1}^d \sum_{v=1}^m \sum_{u_1,u_2=1}^d \sum_{l_1=1}^m E\left|\int_{\kappa(n,t)}^t \frac{\partial^2 \sigma^{(k,v)}(x_r^n)}{\partial x^{u_1} \partial x^{u_2}} \tilde{\sigma}^{(u_1,l_1)}(x_{\kappa(n,r)}^n) \tilde{\sigma}^{(u_2,l_1)}(x_{\kappa(n,r)}^n) dr\right|^q \\
&+ K \sum_{u,k=1}^d \sum_{v=1}^m \sum_{u=1}^d \sum_{j=1}^m E\left|\int_{\kappa(n,t)}^t \sigma^{(u,j)}(x_{\kappa(n,r)}^n) \left\{ \frac{\partial \sigma^{(k,v)}(x_r^n)}{\partial x^u} - \frac{\partial \sigma^{(k,v)}(x_{\kappa(n,r)}^n)}{\partial x^u} \right\} dw_r^j\right|^q \\
&+ K \sum_{u,k=1}^d \sum_{v=1}^m \sum_{u=1}^d \sum_{j=1}^m E\left|\int_{\kappa(n,t)}^t \frac{\partial \sigma^{(k,v)}(x_r^n)}{\partial x^u} \sigma_1^{(u,j)}(x_{\kappa(n,r)}^n) dw_r^j\right|^q
\end{aligned}$$

and the application of Remarks [4, 5, 12, 7], Assumption A-3, Lemmas [4, 19] and Corollary 1 completes the proof. \square

Lemma 22. *Let Assumptions A-1 to A-3 (with $\gamma \equiv 0$) and A-5 hold, then*

$$E \int_0^t |e_s^n|^{q-2} e_s^n (b(x_s^n) - b(x_{\kappa(n,s)}^n)) ds \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds + K n^{-q}$$

for any $t \in [0, T]$.

Proof. The proof of the lemma is done for any $q \geq 4$ since the results for the case $q = 2$ is a direct consequence of Lemma 18 when $\gamma \equiv 0$. As before, one uses Ito's formula to write $b^k(x_s^n) - b^k(x_{\kappa(n,s)}^n)$ and then obtains the following,

$$\begin{aligned}
E \int_0^t |e_s^n|^{q-2} e_s^n (b(x_s^n) - b(x_{\kappa(n,s)}^n)) ds &= \sum_{k=1}^d E \int_0^t |e_s^n|^{q-2} e_s^{n,k} (b^k(x_s^n) - b^k(x_{\kappa(n,s)}^n)) ds \\
&= \sum_{k,i=1}^d E \int_0^t |e_s^n|^{q-2} e_s^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{b}^{n,i}(x_{\kappa(n,r)}^n) dr ds \\
&+ \frac{1}{2} \sum_{k,i,j=1}^d E \int_0^t |e_s^n|^{q-2} e_s^{n,k} \int_{\kappa(n,s)}^s \frac{\partial^2 b^k(x_r^n)}{\partial x^i \partial x^j} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) \tilde{\sigma}^{(j,l)}(x_{\kappa(n,r)}^n) dr ds \\
&+ \sum_{k,i=1}^d E \int_0^t |e_s^n|^{q-2} e_s^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&=: T_1 + T_2 + T_3.
\end{aligned} \tag{45}$$

By using Schwarz and Young's inequalities, T_1 can be estimated as

$$\begin{aligned}
T_1 &:= \sum_{k,i=1}^d E \int_0^t |e_s^n|^{q-2} e_s^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{b}^{n,i}(x_{\kappa(n,r)}^n) dr ds \\
&\leq K E \int_0^t |e_s^n|^q ds + K \sum_{k,i=1}^d E \int_0^t \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{b}^{n,i}(x_{\kappa(n,r)}^n) dr \right|^q ds
\end{aligned}$$

which on the application of Remarks [12, 6, 7] and Lemma 11 implies

$$T_1 \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds + K n^{-q}. \quad (46)$$

By using Schwarz, Young's and Hölder's inequalities, Remark 6, Corollary 1 and Lemma 11, T_2 can be estimated as,

$$\begin{aligned} T_2 &:= \frac{1}{2} \sum_{k,i,j=1}^d E \int_0^t |e_s^n|^{q-2} e_s^{n,k} \int_{\kappa(n,s)}^s \frac{\partial^2 b^k(x_r^n)}{\partial x^i \partial x^j} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) \tilde{\sigma}^{(j,l)}(x_{\kappa(n,r)}^n) dr ds \\ &\leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds + K n^{-q}. \end{aligned} \quad (47)$$

For the last term of equation (45), one uses Itô's formula to obtain the following,

$$\begin{aligned} |e_s^n|^{q-2} e_s^{n,k} &= |e_{\kappa(n,s)}^n|^{q-2} e_{\kappa(n,s)}^{n,k} + \int_{\kappa(n,s)}^s |e_r^n|^{q-2} (b^k(x_r) - \tilde{b}^{n,k}(x_{\kappa(n,r)}^n)) dr \\ &\quad + \int_{\kappa(n,s)}^s |e_r^n|^{q-2} \sum_{v=1}^m (\sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n)) dw_r^v \\ &\quad + (q-2) \int_{\kappa(n,s)}^s e_r^{n,k} |e_r^n|^{q-4} e_r^n (b(x_r) - \tilde{b}^n(x_{\kappa(n,r)}^n)) dr \\ &\quad + (q-2) \int_{\kappa(n,s)}^s e_r^{n,k} |e_r^n|^{q-4} e_r^n (\sigma(x_r) - \tilde{\sigma}(x_{\kappa(n,r)}^n)) dw_r \\ &\quad + \frac{(q-2)(q-4)}{2} \int_{\kappa(n,s)}^s e_r^{n,k} |e_r^n|^{q-6} |(\sigma(x_r) - \tilde{\sigma}(x_{\kappa(n,r)}^n))^* e_r^n|^2 dr \\ &\quad + \frac{q-2}{2} \int_{\kappa(n,s)}^s e_r^{n,k} |e_r^n|^{q-4} |\sigma(x_r) - \tilde{\sigma}(x_{\kappa(n,r)}^n)|^2 dr \\ &\quad + (q-2) \int_{\kappa(n,s)}^s \sum_{v=1}^m (\sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n)) |e_r^n|^{q-4} \\ &\quad \times \sum_{u=1}^d e_r^{n,u} (\sigma^{(u,v)}(x_r) - \tilde{\sigma}^{(u,v)}(x_{\kappa(n,r)}^n)) dr \end{aligned} \quad (48)$$

almost surely for any $s \in [0, T]$. Hence, on substituting the values from equation (48) in T_3 of equation (45)

and then on using Schwarz inequality, one obtains

$$\begin{aligned}
T_3 &= \sum_{k,i=1}^d E \int_0^t |e_{\kappa(n,s)}^n|^{q-2} e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&+ \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-2} |b(x_r) - \tilde{b}^n(x_{\kappa(n,r)}^n)| dr \\
&\quad \times \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right| ds \\
&+ \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-2} \sum_{v=1}^m (\sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n)) dw_r^v \\
&\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&+ K \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s e_r^{n,k} |e_r^n|^{q-4} e_r^n (\sigma(x_r) - \tilde{\sigma}(x_{\kappa(n,r)}^n)) dw_r \\
&\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&\quad + K \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-3} |\sigma(x_r) - \tilde{\sigma}(x_{\kappa(n,r)}^n)|^2 dr \\
&\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&=: T_{31} + T_{32} + T_{33} + T_{34} + T_{35}.
\end{aligned} \tag{49}$$

Furthermore, for estimating T_{31} , one writes,

$$\begin{aligned}
T_{31} &:= K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t |e_{\kappa(n,s)}^n|^{q-2} e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&= K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t |e_{\kappa(n,s)}^n|^{q-2} e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&\quad + K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t |e_{\kappa(n,s)}^n|^{q-2} e_{\kappa(n,s)}^{n,k} \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_1^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds
\end{aligned}$$

which on using the Young's inequality, Hölder's inequality and an elementary inequality of stochastic inte-

grals as well as by noticing that the first term is zero, one obtains the following,

$$\begin{aligned}
T_{31} &\leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds \\
&\quad + K \sum_{k,i=1}^d \sum_{l=1}^m E \int_0^t \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sigma_1^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^q ds \\
&\leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds \\
&\quad + K n^{-\frac{q}{2}+1} E \int_0^t \int_{\kappa(n,s)}^s (1 + |x_r^n|^\chi)^q |\sigma_1(x_{\kappa(n,r)}^n)|^q dr ds \\
&\leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds \\
&\quad + K n^{-\frac{q}{2}+1} E \int_0^t \int_{\kappa(n,s)}^s (E(1 + |x_r^n|^\chi)^{\frac{qp}{p-q}})^{\frac{p-q}{p}} (E|\sigma_1(x_{\kappa(n,r)}^n)|^p)^{\frac{q}{p}} dr ds
\end{aligned}$$

and then the application of Lemmas [4, 11] yields,

$$T_{31} \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds + K n^{-q}. \quad (50)$$

Also, to estimate T_{32} , one uses the splitting (28) to obtain,

$$\begin{aligned}
T_{32} &:= \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-2} |b(x_r) - \tilde{b}^n(x_{\kappa(n,r)}^n)| dr \\
&\quad \times \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right| ds \\
&\leq K \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s (n^{\frac{1}{q}} |e_r^n|)^{q-1} (1 + |x_r^n|^\chi + |x_{\kappa(n,r)}^n|^\chi) dr \\
&\quad \times n^{-\frac{q-1}{q}} \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right| ds \\
&\quad + K \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s (n^{\frac{1}{q}} |e_r^n|)^{q-2} (1 + |x_r^n|^\chi + |x_{\kappa(n,r)}^n|^\chi) |x_r^n - x_{\kappa(n,r)}^n| dr \\
&\quad \times n^{-\frac{q-2}{q}} \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right| ds \\
&\quad + K \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s (n^{\frac{1}{q}} |e_r^n|)^{q-2} |b(x_{\kappa(n,r)}^n) - \tilde{b}^n(x_{\kappa(n,r)}^n)| dr \\
&\quad \times n^{-\frac{q-2}{q}} \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right| ds.
\end{aligned}$$

Due to Young's inequality and Remark 7,

$$\begin{aligned}
T_{32} &\leq KE \int_0^t n \int_{\kappa(n,s)}^s |e_r^n|^q dr ds + Kn^{-q+1} \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s (1 + |x_r|^\chi + |x_r^n|^\chi)^q dr \\
&\quad \times \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^q ds \\
&\quad + Kn^{-\frac{q-2}{2}} \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s (1 + |x_r^n|^\chi + |x_{\kappa(n,r)}^n|^\chi)^{\frac{q}{2}} |x_r^n - x_{\kappa(n,r)}^n|^{\frac{q}{2}} dr \\
&\quad \times \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^{\frac{q}{2}} ds \\
&\quad + Kn^{-q+1} \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s (1 + |x_{\kappa(n,r)}^n|^{\chi+2})^q dr \\
&\quad \times \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^{\frac{q}{2}} ds.
\end{aligned}$$

which on using Hölder's inequality yields,

$$\begin{aligned}
T_{32} &\leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds + Kn^{-q+1} \sum_{k,i=1}^d \int_0^t \left[E \left(\int_{\kappa(n,s)}^s (1 + |x_r|^\chi + |x_r^n|^\chi)^q dr \right)^2 \right]^{\frac{1}{2}} \\
&\quad \times \left[E \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^{2q} \right]^{\frac{1}{2}} ds \\
&\quad + Kn^{-\frac{q-2}{2}} \sum_{k,i=1}^d \int_0^t \left[E \left(\int_{\kappa(n,s)}^s (1 + |x_r^n|^\chi + |x_{\kappa(n,r)}^n|^\chi)^{\frac{q}{2}} |x_r^n - x_{\kappa(n,r)}^n|^{\frac{q}{2}} dr \right)^2 \right]^{\frac{1}{2}} \\
&\quad \times \left[E \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^{q^2} \right]^{\frac{1}{2}} ds \\
&\quad + Kn^{-q+1} \sum_{k,i=1}^d \int_0^t \left[E \left(\int_{\kappa(n,s)}^s (1 + |x_{\kappa(n,r)}^n|^{\chi+2})^q dr \right)^2 \right]^{\frac{1}{2}} \\
&\quad \times \left[E \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^q \right]^{\frac{1}{2}} ds
\end{aligned}$$

and then due to Remark 6, one obtains

$$\begin{aligned}
T_{32} &\leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds + K n^{-q+1} \sum_{k,i=1}^d \int_0^t \left[n^{-1} E \int_{\kappa(n,s)}^s (1 + |x_r|^\chi + |x_r^n|^\chi)^{2q} dr \right]^{\frac{1}{2}} \\
&\quad \times \left[n^{-q+1} E \int_{\kappa(n,s)}^s (1 + |x_r^n|^{\chi+1})^{2q} |\tilde{\sigma}(x_{\kappa(n,r)}^n)|^{2q} dr \right]^{\frac{1}{2}} ds \\
&\quad + K n^{-\frac{q-2}{2}} \int_0^t \left[n^{-1} E \int_{\kappa(n,s)}^s (1 + |x_r|^\chi + |x_{\kappa(n,r)}^n|^\chi)^q |x_r^n - x_{\kappa(n,r)}^n|^q dr \right]^{\frac{1}{2}} \\
&\quad \times \left[n^{-\frac{q}{2}+1} E \int_{\kappa(n,s)}^s (1 + |x_r^n|^{\chi+1})^q |\tilde{\sigma}(x_{\kappa(n,r)}^n)|^q dr \right]^{\frac{1}{2}} ds \\
&\quad + K n^{-q+1} \int_0^t \left[n^{-1} E \int_{\kappa(n,s)}^s (1 + |x_{\kappa(n,r)}^n|^{\chi+2})^{2q} dr \right]^{\frac{1}{2}} \\
&\quad \times \left[n^{-\frac{q}{2}+1} E \int_{\kappa(n,s)}^s (1 + |x_r^n|^{\chi+1})^q |\tilde{\sigma}(x_{\kappa(n,r)}^n)|^q dr \right]^{\frac{1}{2}} ds.
\end{aligned}$$

Thus, by the application of Hölder's inequality and Lemmas [2, 11, 19] and Corollary 1, one obtains,

$$T_{32} \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds + K n^{-q}. \quad (51)$$

For estimating T_{33} , one uses the splitting (30) to write

$$\begin{aligned}
T_{33} &:= \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-2} \sum_{v=1}^m (\sigma^{(k,v)}(x_r) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n)) dw_r^v \\
&\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&= \sum_{k,i=1}^d \sum_{v,l=1}^m E \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-2} (\sigma^{(k,v)}(x_r) - \sigma^{(k,v)}(x_r^n)) dw_r^v \\
&\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&\quad + \sum_{k,i=1}^d \sum_{v,l=1}^m E \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-2} (\sigma^{(k,v)}(x_r^n) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n)) dw_r^v \\
&\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&= \sum_{k,i=1}^d \sum_{v=1}^m E \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-2} (\sigma^{(k,v)}(x_r) - \sigma^{(k,v)}(x_r^n)) \\
&\quad \times \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,v)}(x_{\kappa(n,r)}^n) dr ds \\
&\quad + \sum_{k,i=1}^d \sum_{v,l=1}^m E \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-2} (\sigma^{(k,v)}(x_r^n) - \tilde{\sigma}^{(k,v)}(x_{\kappa(n,r)}^n)) \\
&\quad \times \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,v)}(x_{\kappa(n,r)}^n) dr ds
\end{aligned}$$

and then Assumption A-2 and Remark 7 give

$$\begin{aligned}
T_{33} &\leq KE \int_0^t \int_{\kappa(n,s)}^s (n^{\frac{1}{q}} |e_r^n|)^{q-1} n^{-\frac{q-1}{q}} (1 + |x_r^n|^{\chi+1}) |\tilde{\sigma}(x_{\kappa(n,r)}^n)| dr ds \\
&\quad + KE \int_0^t \int_{\kappa(n,s)}^s (n^{\frac{1}{q}} |e_r^n|)^{q-2} n^{-\frac{q-2}{q}} |\sigma(x_r^n) - \tilde{\sigma}(x_{\kappa(n,r)}^n)| (1 + |x_r^n|^{\chi+1}) |\tilde{\sigma}(x_{\kappa(n,r)}^n)| dr ds
\end{aligned} \tag{52}$$

which due to Young's inequality and Hölder's inequality yields

$$\begin{aligned}
T_{33} &\leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds \\
&\quad + Kn^{-q+1} \int_0^t \int_{\kappa(n,s)}^s (E(1 + |x_r^n|^{\chi+1})^{2q} E |\tilde{\sigma}(x_{\kappa(n,r)}^n)|^{2q})^{\frac{1}{2}} dr ds \\
&\quad + Kn^{-\frac{q}{2}+1} \int_0^t \int_{\kappa(n,s)}^s (E |\sigma(x_r^n) - \tilde{\sigma}(x_{\kappa(n,r)}^n)|^q E(1 + |x_r^n|^{\chi+1})^q |\tilde{\sigma}(x_{\kappa(n,r)}^n)|^q)^{\frac{1}{2}} dr ds.
\end{aligned}$$

and on the application of Lemmas [11, 21] and Corollary 1, one obtains

$$T_{33} \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds + Kn^{-q}. \tag{53}$$

To estimate T_{34} , one observes that

$$\begin{aligned}
T_{34} &:= K \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s e_r^{n,k} |e_r^n|^{q-4} e_r^n (\sigma(x_r) - \tilde{\sigma}(x_{\kappa(n,r)}^n)) dw_r \\
&\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&= K \sum_{k,i,u=1}^d \sum_{v,l=1}^m E \int_0^t \int_{\kappa(n,s)}^s e_r^{n,k} |e_r^n|^{q-4} e_r^{n,u} (\sigma^{(u,v)}(x_r) - \tilde{\sigma}^{(u,v)}(x_{\kappa(n,r)}^n)) dw_r^v \\
&\quad \times \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l ds \\
&= K \sum_{k,i,u=1}^d \sum_{v=1}^m E \int_0^t \int_{\kappa(n,s)}^s e_r^{n,k} |e_r^n|^{q-4} e_r^{n,u} (\sigma^{(u,v)}(x_r) - \tilde{\sigma}^{(u,v)}(x_{\kappa(n,r)}^n)) \\
&\quad \times \frac{\partial b^k(x_r^n)}{\partial x^i} \tilde{\sigma}^{(i,v)}(x_{\kappa(n,r)}^n) dr ds
\end{aligned}$$

and then due to splitting (30) along with Remark 6, one obtains,

$$\begin{aligned}
T_{34} &\leq KE \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-1} (1 + |x_r^n|^{\chi+1}) |\tilde{\sigma}(x_{\kappa(n,r)}^n)| dr ds \\
&\quad + KE \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-2} |\sigma(x_r^n) - \tilde{\sigma}(x_{\kappa(n,r)}^n)| (1 + |x_r^n|^{\chi+1}) |\tilde{\sigma}(x_{\kappa(n,r)}^n)| dr ds
\end{aligned}$$

which is exactly the same as T_{33} given in equation (52). Thus, by proceeding exactly the same, one obtains,

$$T_{34} \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds + Kn^{-q}. \tag{54}$$

Moreover, for estimating T_{35} , one again uses the splitting (30) to obtain,

$$\begin{aligned}
T_{35} &:= K \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-3} |\sigma(x_r) - \tilde{\sigma}(x_{\kappa(n,r)}^n)|^2 dr \\
&\quad \times \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right| ds \\
&\leq K \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-1} dr \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right| ds \\
&\quad + K \sum_{k,i=1}^d E \int_0^t \int_{\kappa(n,s)}^s |e_r^n|^{q-3} |\sigma(x_r^n) - \tilde{\sigma}(x_{\kappa(n,r)}^n)|^2 dr \\
&\quad \times \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right| ds
\end{aligned}$$

which due to Hölder's inequality implies

$$\begin{aligned}
T_{35} &\leq K \sum_{k,i=1}^d \int_0^t \left(E \left[\int_{\kappa(n,s)}^s |e_r^n|^{q-1} dr \right]^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \\
&\quad \times \left(E \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^q \right)^{\frac{1}{q}} ds \\
&\quad + K \sum_{k,i=1}^d \int_0^t \left(E \left[\int_{\kappa(n,s)}^s |e_r^n|^{q-3} |\sigma(x_r^n) - \tilde{\sigma}(x_{\kappa(n,r)}^n)|^2 dr \right]^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \\
&\quad \times \left(E \left| \int_{\kappa(n,s)}^s \frac{\partial b^k(x_r^n)}{\partial x^i} \sum_{l=1}^m \tilde{\sigma}^{(i,l)}(x_{\kappa(n,r)}^n) dw_r^l \right|^q \right)^{\frac{1}{q}} ds
\end{aligned}$$

and then an elementary inequality of stochastic integrals along with Hölder's inequality yields

$$\begin{aligned}
T_{35} &\leq K n^{-\frac{1}{2}} \int_0^t \left(n^{-\frac{q}{q-1}+1} E \int_{\kappa(n,s)}^s |e_r^n|^q dr \right)^{\frac{q-1}{q}} ds \\
&\quad + K n^{-\frac{1}{2}} \int_0^t \left(n^{-\frac{q}{q-1}+1} E \int_{\kappa(n,s)}^s |e_r^n|^{\frac{(q-3)q}{q-1}} |\sigma(x_r^n) - \tilde{\sigma}(x_{\kappa(n,r)}^n)|^{\frac{2q}{q-1}} dr \right)^{\frac{q-1}{q}} ds \\
&\leq K n^{-\frac{3}{2}} \int_0^t \left(\sup_{0 \leq r \leq s} E |e_r^n|^q \right)^{\frac{q-1}{q}} ds \\
&\quad + K n^{-\frac{1}{2}} \int_0^t \left(n^{-\frac{q}{q-1}+1} \int_{\kappa(n,s)}^s (E |e_r^n|^q)^{\frac{q-3}{q-1}} (E |\sigma(x_r^n) - \tilde{\sigma}(x_{\kappa(n,r)}^n)|^q)^{\frac{2}{q-1}} dr \right)^{\frac{q-1}{q}} ds \\
&\leq K n^{-\frac{3}{2}} \int_0^t \left(\sup_{0 \leq r \leq s} E |e_r^n|^q \right)^{\frac{q-1}{q}} ds + K n^{-\frac{7}{2}} \int_0^t \left(\sup_{0 \leq r \leq s} E |e_r^n|^q \right)^{\frac{q-3}{q}} ds.
\end{aligned}$$

Thus, by Young's inequality,

$$T_{35} \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds + K n^{-\frac{3q}{2}} + K n^{-\frac{7q}{6}}. \quad (55)$$

Hence, on substituting the estimates from (50), (51), (53), (54) and (55) in (49), one obtains

$$T_3 \leq K \int_0^t \sup_{0 \leq r \leq s} E |e_r^n|^q ds + K n^{-q}. \quad (56)$$

Finally, on combining the estimates from (46), (47) and (56) in (45), one completes the proof. \square

Proof of Theorem 2. Let us take $e_t^n := x_t - x_t^n$ i.e.,

$$e_t^n = \int_0^t (b(x_s) - \tilde{b}^n(x_{\kappa(n,s)}^n)) ds + \int_0^t (\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}^n)) dw_s$$

for any $t \in [0, T]$. By Itô's formula, for any $q \geq 2$,

$$\begin{aligned} |e_t^n|^q &= q \int_0^t |e_s^n|^{q-2} e_s^n (b(x_s) - \tilde{b}^n(x_{\kappa(n,s)}^n)) ds \\ &\quad + q \int_0^t |e_s^n|^{q-2} e_s^n (\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}^n)) dw_s \\ &\quad + \frac{q(q-2)}{2} \int_0^t |e_s^n|^{q-4} |(\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}^n))^* e_s^n|^2 ds \\ &\quad + \frac{q}{2} \int_0^t |e_s^n|^{q-2} |\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 ds \end{aligned}$$

almost surely for any $t \in [0, T]$, which on taking expectation and using Schwarz inequality implies

$$\begin{aligned} E|e_t^n|^q &\leq qE \int_0^t |e_s^n|^{q-2} e_s^n (b(x_s) - \tilde{b}^n(x_{\kappa(n,s)}^n)) ds \\ &\quad + \frac{q(q-1)}{q} \int_0^t |e_s^n|^{q-2} |\sigma(x_s) - \tilde{\sigma}(x_{\kappa(n,s)}^n)|^2 ds. \end{aligned} \quad (57)$$

Thus, by using the estimates from (37) and (39), one obtains,

$$\begin{aligned} E|e_t^n|^q &\leq KE \int_0^t |e_s^n|^q ds + KE \int_0^t |e_s^n|^{q-2} e_s^n (b(x_s^n) - \tilde{b}^n(x_{\kappa(n,s)}^n)) ds \\ &\quad + KE \int_0^t |b(x_{\kappa(n,s)}^n) - \tilde{b}^n(x_{\kappa(n,s)}^n)|^q ds + KE \int_0^t |\sigma(x_s^n) - \tilde{\sigma}(x_{\kappa(n,s)}^n)|^q ds \end{aligned} \quad (58)$$

for any $t \in [0, T]$. Further, on the application of Lemmas [20, 21, 22] gives

$$\sup_{0 \leq t \leq u} E|e_t^n|^q \leq K \int_0^u \sup_{0 \leq r \leq s} E|e_r^n|^q ds + Kn^{-q}$$

for any $u \in [0, T]$ and hence Gronwall's lemma completes the proof. \square

Proof of Theorem 3. This follows due to Theorem 2 and Lemma 3. \square

6 Numerical Simulation

In this section, we demonstrate theoretical results obtained in this article with the help of some examples. In order to implement the scheme (3) of SDE (1) and scheme (43) of SDE (42), one requires commutative conditions on diffusion and jump coefficients. One can refer to [14] and [17] for details. For the purpose of this section, we consider one-dimensional SDEs defined on interval $[0, 1]$, which is partitioned into 2^n sub-intervals of equal length (step-size) $h = 2^{-n}$ for some $n \in \mathbb{N}$. In what follows, x_{lh}^h denotes the tamed Milstein scheme at lh -th grid point and $\Delta w_{lh} := w_{(l+1)h} - w_{lh}$ for $l = 0, \dots, 2^n$.

Example 6.1. Let us consider one-dimensional SDE defined by,

$$dx_t = (x_t - x_t^5)dt + x_t dw_t \quad (59)$$

for any $t \in [0, 1]$ with $x_0 = 1$. The tamed Milstein scheme of SDE (59) at $(l+1)h$ -th grid is given by

$$x_{(l+1)h}^h = x_{lh}^h + \frac{x_{lh}^h - (x_{lh}^h)^5}{1 + h|x_{lh}^h - (x_{lh}^h)^5|^2}h + x_{lh}^h \Delta w_{lh} + \frac{1}{2}x_{lh}^h \{(\Delta w_{lh})^2 - h\} \quad (60)$$

for $l = 0, \dots, 2^n - 1$ with $h = 2^{-n}$. Here, the above scheme with $h = 2^{-21}$ is taken as true solution of SDE (59). From Table 1 and Figure 1, one can notice that \mathcal{L}^q convergence rates of scheme (60) are approximately 1.0 for all $q = 1, \dots, 5$, which are consistent with our theoretical findings. The number of paths considered is 60,000.

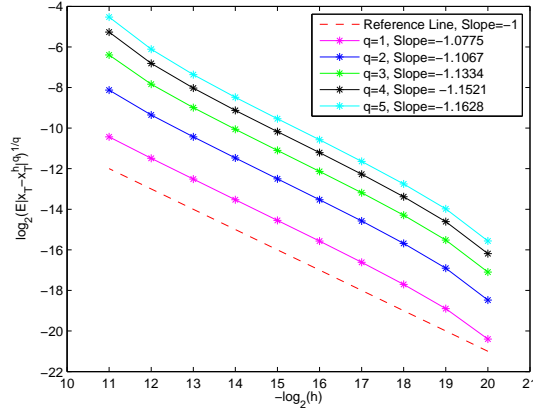


Figure 1: \mathcal{L}^q -convergence rate of tamed Milstein scheme (60) of SDE (59)

$h = 2^{-n}$	$E x_T - x_T^h $	$\sqrt{E x_T - x_T^h ^2}$	$\sqrt[3]{E x_T - x_T^h ^3}$	$\sqrt[4]{E x_T - x_T^h ^4}$	$\sqrt[5]{E x_T - x_T^h ^5}$
2^{-20}	0.0000007223	0.0000027396	0.0000071338	0.0000133872	0.0000206505
2^{-19}	0.0000020488	0.0000081546	0.0000213011	0.0000400356	0.0000618759
2^{-18}	0.0000046829	0.0000190372	0.0000498670	0.0000937473	0.0001448460
2^{-17}	0.0000099526	0.0000408359	0.0001072262	0.0002022094	0.0003133946
2^{-16}	0.0000205589	0.0000844630	0.0002227601	0.0004218232	0.0006555927
2^{-15}	0.0000417394	0.0001723833	0.0004554010	0.0008642163	0.0013460486
2^{-14}	0.0000843948	0.0003519474	0.0009360518	0.0017870537	0.0027962853
2^{-13}	0.0001710052	0.0007232200	0.0019649384	0.0038259755	0.0060684654
2^{-12}	0.0003479789	0.0015293072	0.0043796657	0.0089031484	0.0144907359
2^{-11}	0.0007231189	0.0035802581	0.0118774764	0.0259649292	0.0432914310

Table 1: Tamed Milstein scheme (60) of SDE (59)

□

Example 6.2. Let us now consider the following one-dimensional SDE,

$$dx_t = -0.10x_t^3 dt + x_t dw_t + \int_{\mathbb{R}} x_t z \tilde{N}(dt, dz) \quad (61)$$

for any $t \in [0, 1]$ with $x_0 = 1$. The tamed Milstein scheme of SDE (61) is given by,

$$\begin{aligned}
x_{(l+1)h}^h = & x_{lh}^h + \frac{-0.10(x_{lh}^h)^3}{1 + h|0.10(x_{lh}^h)^3|^2}h + x_{lh}^h \Delta w_{lh} \\
& + x_{lh}^h \sum_{i=N((lh, \mathbb{R})+1)}^{N((l+1)h, \mathbb{R})} z_i + \frac{1}{2}x_{lh}^h \{(\Delta w_{lh})^2 - h\} \\
& + x_{lh}^h \sum_{i=N((lh, \mathbb{R})+1)}^{N((l+1)h, \mathbb{R})} z_i(w_{(l+1)h} - w_{\tau_i}) + x_{lh}^h \sum_{i=N((lh, \mathbb{R})+1)}^{N((l+1)h, \mathbb{R})} z_i(w_{\tau_i} - w_{lh}) \\
& + x_{lh}^h \sum_{j=N((lh, \mathbb{R})+1)}^{N((l+1)h, \mathbb{R})} \sum_{i=N((lh, \mathbb{R})+1)}^{N(\tau_j, \mathbb{R})} z_i z_j
\end{aligned} \tag{62}$$

for $l = 0, \dots, 2^n - 1$. Notice that $N(t, \mathbb{R})$ is a Poisson process with (jump) intensity λ .

The case $\lambda = 3$ in Table 2a and its corresponding plot in Figure 2a are based on 60,000 trajectories while case $\lambda = 5$ of Table 2a and its corresponding plot in Figure 2b are based on 360,000 paths. In both the cases, the mark random variables z_i s (jump-sizes) are assumed to follow normal distribution with mean 0 and variance 0.125.

Similarly, both the cases $\lambda = 3$ and $\lambda = 5$ in Table 2b and their corresponding plots in Figure 2c and Figure 2d are based on 240,000 simulations. Here, the mark random variables z_i s are assumed to follow uniform distribution on $[-1/4, 1/4]$.

$h = 2^{-n}$	$\sqrt{E x_T - x_T^h ^2}$	
	$\lambda = 3.0$	$\lambda = 5.0$
2^{-20}	0.00067484	0.00621555
2^{-19}	0.00204889	0.02203719
2^{-18}	0.00515874	0.06003098
2^{-17}	0.01321011	0.27830910
2^{-16}	0.03146860	0.45542612
2^{-15}	0.06349005	0.66561201
2^{-14}	0.16459365	0.85082102
2^{-13}	0.27426757	1.55620505
2^{-12}	0.40437133	2.07850380
2^{-11}	0.57251083	2.41922833

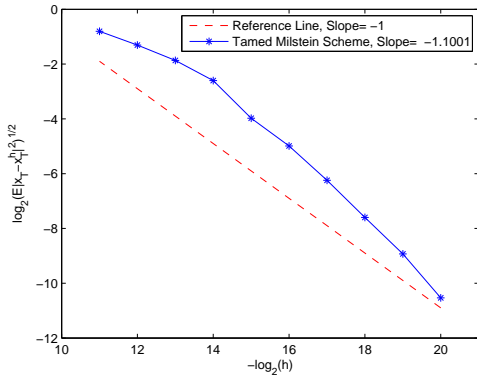
(a) Mark is normal with mean 0 and variance 0.125.

$h = 2^{-n}$	$\sqrt{E x_T - x_T^h ^2}$	
	$\lambda = 3.0$	$\lambda = 5.0$
2^{-20}	0.00004365	0.00005260
2^{-19}	0.00011286	0.00013058
2^{-18}	0.00025133	0.00028420
2^{-17}	0.00054134	0.00059787
2^{-16}	0.00112643	0.00125615
2^{-15}	0.00242178	0.00272857
2^{-14}	0.00563126	0.00673629
2^{-13}	0.01497166	0.01747566
2^{-12}	0.03745749	0.03448135
2^{-11}	0.07302734	0.07900926

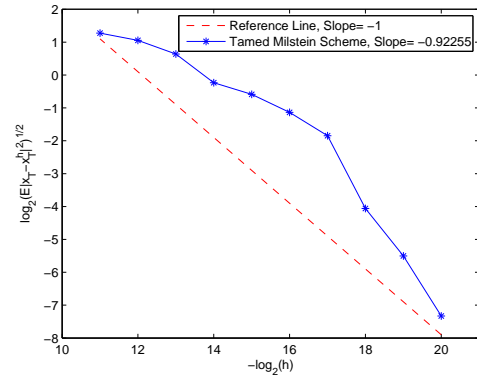
(b) Mark is uniform on $[-1/4, 1/4]$.

Table 2: Tamed Milstein scheme (62) of SDE (61).

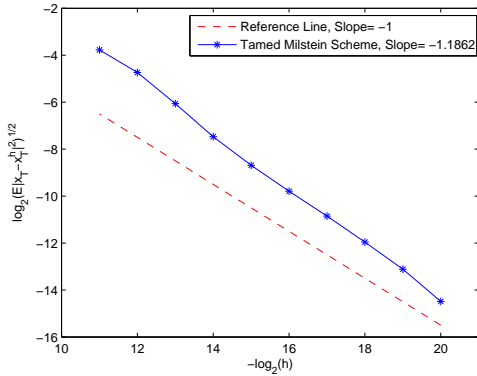
It is evident from plots in Figure 2 that rate of convergence of the tamed Milstein scheme of SDE driven by Lévy noise depends significantly on the distribution of mark random variable and the jump intensity. \square



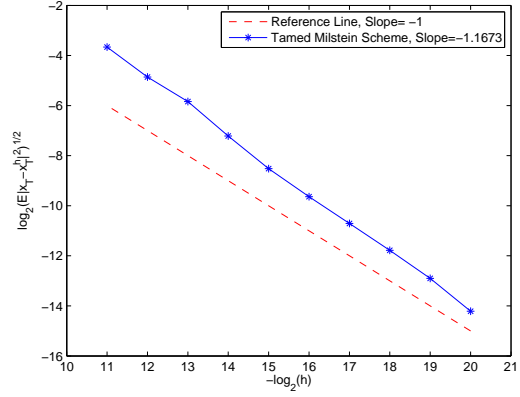
(a) $\lambda = 3$ and mark is normal with mean 0 and variance 0.125



(b) $\lambda = 5$ and mark is normal with mean 0 and variance 0.125



(c) $\lambda = 3$ and mark is uniform on $[-1/4, 1/4]$



(d) $\lambda = 5$ and mark is uniform on $[-1/4, 1/4]$

Figure 2: \mathcal{L}^2 -convergence rate of tamed Milstein scheme (62) of SDE (61).

Appendix A

In this section, one discusses about implementing the scheme (3) on computer. For the purpose of simplicity, let us assume that

$$\int_Z \gamma(x, z) \nu(dz) = 0 \quad (63)$$

for any $x \in \mathbb{R}^d$. Thus, SDE (1) becomes

$$x_t = \xi + \int_0^t b(x_s) ds + \int_0^t \sigma(x_s) dw_s + \int_0^t \int_Z \gamma(x_s, z) N(ds, dz) \quad (64)$$

for any $t \in [0, T]$. One can partition the interval $[0, T]$ into H subintervals of length h such that $Hh = T$, in particular $H = 2^n T$ and $h = 2^{-n}$ for any $n \in \mathbb{N}$. For simplicity, one can take $T = 1$. As a consequence,

the tamed Milstein scheme (3) at the $(l+1)h$ -th grid can be written as,

$$\begin{aligned}
x_{(l+1)h}^h &= x_{lh}^h + \tilde{b}^n(x_{lh}^h)h + \sum_{k=1}^m \sigma^{(k)}(x_{lh}^h) \Delta w_{lh}^k + \int_{lh}^{(l+1)h} \int_Z \gamma(x_{lh}^h, z_2) N(ds, dz_2) \\
&+ \sum_{j,k=1}^m \sum_{u=1}^d \int_{lh}^{(l+1)h} \int_{lh}^s \sigma^{(u,j)}(x_{lh}^h) \frac{\partial \sigma^{(k)}(x_{lh}^h)}{\partial x^u} dw_r^j dw_s^k \\
&+ \sum_{j=1}^m \int_{lh}^{(l+1)h} \int_{lh}^s \int_Z \{ \sigma^{(j)}(x_{lh}^h + \gamma(x_{lh}^h, z_1)) - \sigma^{(j)}(x_{lh}^h) \} N(dr, dz_1) dw_s^j \\
&+ \sum_{j=1}^m \sum_{u=1}^d \int_{lh}^{(l+1)h} \int_{lh}^s \int_Z \frac{\partial \gamma(x_{lh}^h, z_2)}{\partial x^u} \sigma^{(u,j)}(x_{lh}^h) dw_r^j N(ds, dz_2) \\
&+ \int_{lh}^{(l+1)h} \int_Z \int_{lh}^s \int_Z \{ \gamma(x_{lh}^h + \gamma(x_{lh}^h, z_1), z_2) - \gamma(x_{lh}^h, z_2) \} N(dr, dz_1) N(ds, dz_2) \quad (65)
\end{aligned}$$

for any $l = 0, \dots, 2^n - 1$, where $h = 2^{-n}$ for $n \in \mathbb{N}$. The drift coefficient \tilde{b}^n of scheme (65) is defined in equation (2). The one-dimensional case of scheme (65) is given by

$$\begin{aligned}
x_{lh+h}^h &= x_{lh}^h + \tilde{b}^n(x_{lh}^h)h + \sigma(x_{lh}^h) \Delta w_{lh} + \sum_{i=N(lh,Z)+1}^{N(lh+h,Z)} \gamma(x_{lh}^h, z_i) \\
&+ \frac{1}{2} \sigma(x_{lh}^h) \frac{\partial \sigma(x_{lh}^h)}{\partial x^u} \{ (\Delta w_{lh})^2 - h \} \\
&+ \sum_{i=N(lh,Z)+1}^{N(lh+h,Z)} \{ \sigma(x_{lh}^h + \gamma(x_{lh}^h, z_i)) - \sigma(x_{lh}^h) \} (w_{lh+h} - w_{\tau_i}) \\
&+ \sigma(x_{lh}^h) \sum_{i=N(lh,Z)+1}^{N(lh+h,Z)} \frac{\partial \gamma(x_{lh}^h, z_i)}{\partial x^u} (w_{\tau_i} - w_{lh}) \\
&+ \sum_{j=N(lh,Z)+1}^{N(lh+h,Z)} \sum_{i=N(lh,Z)+1}^{N(\tau_j,Z)} \{ \gamma(x_{lh}^h + \gamma(x_{lh}^h, z_i), z_j) - \gamma(x_{lh}^h, z_j) \} \quad (66)
\end{aligned}$$

for $l = 0, \dots, 2^n - 1$. The multi-dimensional case of scheme (65) with mark-dependent jump coefficient is computationally difficult to simulate because one requires to keep track of the jump times in each sub-interval. However, the mark-independent case i.e. when $\gamma(x, z) = \gamma(x)$ for any $x \in \mathbb{R}^d$ and $z \in Z$, can be simplified by imposing diffusion and jump commutative conditions as respectively given below,

$$\sum_{u=1}^d \sigma^{(u,j)}(x) \frac{\partial \sigma^{(i,k)}(x)}{\partial x^u} = \sum_{u=1}^d \sigma^{(u,k)}(x) \frac{\partial \sigma^{(i,j)}(x)}{\partial x^u} \quad (67)$$

for any $k, j = 1, \dots, m$, $i = 1, \dots, d$ and $x \in \mathbb{R}^d$, and

$$\sigma^{(k,j)}(x + \gamma(x)) - \sigma^{(k,j)}(x) = \sum_{u=1}^d \frac{\partial \gamma^k(x)}{\partial x^u} \sigma^{(u,j)}(x) \quad (68)$$

for any $x \in \mathbb{R}^d$, $k = 1, \dots, d$ and $j = 1, \dots, m$. A detailed discussion on commutative conditions can be found in [17]. Thus, the tamed Milstein scheme (65) of SDE (64) with diffusion and jump coefficients satisfying

commutative conditions (67) and (68), can be written as

$$\begin{aligned}
x_{lh+h}^h &= x_{lh}^h + \tilde{b}^n(x_{lh}^h)h + \sum_{k=1}^m \sigma^{(k)}(x_{lh}^h) \Delta w_{lh}^k + \gamma(x_{lh}^h) \Delta N(lh, Z) \\
&+ \frac{1}{2} \sum_{j,k=1}^m \sum_{u=1}^d \sigma^{(u,j)}(x_{lh}^h) \frac{\partial \sigma^{(k)}(x_{lh}^h)}{\partial x^u} \{ \Delta w_{lh}^j \Delta w_{lh}^k - h I_{\{j=k\}} \} \\
&+ \sum_{j=1}^m \{ \sigma^{(j)}(x_{lh}^h + \gamma(x_{lh}^h)) - \sigma^{(j)}(x_{lh}^h) \} \Delta N(lh, Z) \Delta w_{lh}^j \\
&+ \frac{1}{2} \{ \gamma(x_{lh}^h + \gamma(x_{lh}^h)) - \gamma(x_{lh}^h) \} \{ (\Delta N(lh, Z))^2 - \Delta N(lh, Z) \}
\end{aligned} \tag{69}$$

where $\Delta N(lh, Z) = N(lh+h, Z) - N(lh, Z)$ (i.e. the number of jumps in $[lh, lh+h]$) for any $l = 0, \dots, 2^n - 1$. Also, notice that when assumption (63) does not hold, one can implement the scheme (65) by replacing the drift coefficient $b^n(x)$ with $b^n(x) - \int_Z \gamma(x, z) \nu(dz)$.

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